

# Cutting Planes for Families Implying Frankl's Conjecture

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**Abstract.** We find previously unknown families which imply Frankl's conjecture for all families that contain them using an algorithmic framework. The conjecture states that for any non-empty union-closed (UC) family there exists an element in at least half of the sets. Poonen's Theorem characterizes the existence of weights which determine whether a given UC family implies the conjecture for all UC families which contain it. An UC family is *Non-Frankl-Complete* (Non-FC), if and only if it does not imply the conjecture in its elements for some UC family that contains it. We design a cutting-plane method that computes the explicit weights which imply the existence conditions of Poonen's Theorem. This method allows us to answer several open questions, including the construction of a counterexample to an eleven-year-old conjecture of Morris about the structure of generators for Non-FC-families.

**Keywords:** Frankl's conjecture, union-closed families, integer programming, cutting-plane method, extremal combinatorics

## 1 Introduction

Frankl's (union-closed sets) conjecture is a celebrated unsolved problem in combinatorics that was recently brought to the attention of a wider audience as a polymath project led by Timothy Gowers [1]. A non-empty finite family of finite sets  $\mathcal{F}$  is union-closed (UC) if and only if for every  $A, B \in \mathcal{F}$  it follows that  $A \cup B \in \mathcal{F}$ . Frankl's conjecture states that for any UC family  $\mathcal{F}$  there exists an element in at least  $|\mathcal{F}|/2$  sets. The problem appears to have little structure – perhaps the very reason why a proof or disproof remains elusive. In this paper we focus on a well-established method employed to attack the problem referred to as *local configurations* in Bruhn and Schaudt [4], namely UC families that imply the conjecture for all UC families which contain them. In other words, these particular UC families always have an element that is frequent enough to imply the conjecture for all UC families that contain them. In this regard, given an UC family  $\mathcal{A}$ , Poonen's Theorem [14] characterizes the necessity of the implication by the existence of weights on the elements of  $\mathcal{A}$  that obey certain inequalities. Define  $[n] := \{1, 2, \dots, n\}$ . A family of sets  $\mathcal{A}$  *covers*  $n$  elements if and only if the union of all sets in  $\mathcal{A}$  is  $[n]$ . Following Vaughan [17], we say that

an UC family of sets  $\mathcal{A}$  which covers  $n$  elements is *Frankl-Complete* (FC), if and only if for every UC family  $\mathcal{F} \supseteq \mathcal{A}$  there exists  $i \in [n]$  frequent enough to satisfy Frankl’s conjecture. An UC family  $\mathcal{A}$  which covers  $n$  elements is *Non-Frankl-Complete* (Non-FC), if and only if there exists an UC family  $\mathcal{F} \supseteq \mathcal{A}$  such that any  $i \in [n]$  is in less than half of the sets of  $\mathcal{F}$ . Non-FC-families are particularly useful in characterizing *minimal* FC-families, i.e., FC-families that do not contain smaller FC-families, and also other objects of interests defined in Morris [13], which help shed light into structural properties of the conjecture. In addition, Non-FC-families yield natural candidates for possible counterexamples. However, on a more positive note, the pressing relevance of FC and Non-FC-families is evident in existing literature: These objects are at the heart of arguments that yield improved bounds for the problem, as seen in Poonen [14], Gao and Yu [9], Morris [13], Marković [12], Bošnjak and Marković [2], and finally Vučković and Živković [20] which features the current best bound on the size of the universe of elements at  $n \leq 12$ . Furthermore, FC-families are used in Bruhn et al. [3] to prove that Frankl’s conjecture holds for subcubic bipartite graphs. Therefore characterizing a considerable number of previously unknown FC and Non-FC-families – the fundamental contribution of this work which consequently helps settle several open questions of interest – is a clear step toward a better understanding of Frankl’s conjecture.

Characterizing *exactly* which UC families are FC and Non-FC is suprisingly difficult, as evinced by the relative dearth of known FC-families despite the past twenty-five years of research on the matter. Indeed, as is typical of objects in mathematics that are not well-understood, efforts on the topic yield more questions than answers. Previous researchers use special structures and stronger than necessary conditions to determine a number of FC-families. In particular, Poonen [14] proves that any UC family which contains three 3-subsets of a 4-set satisfies the conjecture. Vaughan [17], [18], [19] proves that the conjecture holds for any UC family which contains a 5-set and all of its 4-subsets, or ten of the 4-subsets of a 6-set and their unions, or three 3-subsets of a 7-set with a common element. Furthermore, using a heuristic procedure implemented in a computer algebra system, Vaughan identifies potential weight systems for candidate FC-families and then proves through tedious and technical case analysis that a few more UC families are FC. Still, most FC-families Vaughan discovers are not minimal, in the sense that they contain smaller FC-families as shown by subsequent research or results in this paper. Morris [13] is able to characterize new FC-families on six elements and with the help of a computer program exactly characterizes all minimal FC-families on 5 elements. Given a family of sets  $\mathcal{S}$ , we say that  $\mathcal{S}$  *generates* (or is a *generator* of)  $\mathcal{F}$ , denoted by  $\langle \mathcal{S} \rangle := \mathcal{F}$ , if and only if  $\mathcal{F}$  is the smallest UC family that contains  $\mathcal{S}$ . In order to facilitate the combinatorial analysis of FC-families, Morris [13] introduces the following notion. Let  $FC(k, n)$  denote the smallest  $m$  such that any  $m$  of the  $k$ -sets in  $[n]$  generate a FC-family. As proven in Gao and Yu [9],  $FC(k, n)$  is always defined for sufficiently large  $n$  in relation to  $k$ . Consequently, Morris shows that  $FC(3, 5) = 3$ ,  $FC(4, 5) = 5$ ,  $FC(3, 6) = 4$ ,  $7 \leq FC(4, 6) \leq 8$ ,  $FC(3, 7) \leq 6$  and  $FC(4, 7) \leq 18$ .

Such characterizations further facilitate the search for better bounds (or possible counterexamples). Finally, Marić, Živković, and Vučković [11] formalize a combinatorial search in the interactive theorem prover Isabelle/HOL and show that all families containing four 3-subsets of a 7-set are FC-families. Although not explicitly mentioned in their paper, their result implies that  $FC(3, 7) = 4$  by the lower bound on the number of 3-sets of Morris [13]. In summary, previous research has yielded less than *two dozen* exact characterizations of *minimal* generators of FC-families, with roughly a dozen more characterizations of general FC-families. In light of the above, our main contributions in this paper are the following:

- In the appendix we feature roughly *one hundred* (with many more underway at the time of this writing) previously unknown minimal non-isomorphic (under some permutation of  $[n]$ ) generators of FC-families, thus contributing to a significant increase in the state of knowledge, and improving on all previously known results of the kind. Furthermore, we give the *first* known exact characterizations of minimal generators of FC-families on  $8 \leq n \leq 10$ .
- We design the *first general* computational framework that is able to precisely characterize FC or Non-FC-families by using exact integer programming and other redundant verification routines, thus providing an algorithmic roadmap for settling open questions in Morris [13] and Vaughan [18], [19].
- In particular we construct an explicit counterexample to an eleven-year-old conjecture of Morris [13] about the structure of generators for Non-FC-families. Furthermore we answer in the negative two related questions of Vaughan [18] and Morris [13] regarding a simplified method for proving the existence of weights that yield FC-families. We believe these applications best illustrate the reach and potential of our method.

The connection between Frankl’s conjecture and mathematical programming is well-established in Pulaj, Raymond and Theis [15], where the authors derive the equivalence of the problem with an integer program and investigate related conjectures. Furthermore, given an UC family  $\mathcal{A}$ , Poonen’s Theorem yields a constructive proof to determine if  $\mathcal{A}$  is FC or Non-FC in the form of a fractional polytope with a potentially exponential number of constraints. In general, this makes it difficult to explicitly state the conditions which determine whether a given UC family is FC. To overcome this, we design a cutting-plane method that computes the explicit weights which imply Poonen’s existence conditions. In particular, this paves the way toward automated discovery of FC-families by computational integer programming, especially when coupled with an exact rational solver [6] and other verification routines such as the recent work of

Cheung, Gleixner and Steffy [5]. Our current implementation<sup>1</sup> in SCIP 3.2.1 [8] allows us to characterize *any FC-family up to 10 elements* tested so far.

## 2 Cutting Planes for FC-families

As mentioned in section 1, Poonen’s seminal article [14] precisely characterizes existence conditions for FC-families. Poonen’s theorem is at the basis of all subsequent approaches for classifying FC-families, which in turn form an integral part of finding, as mention already, the best current bounds of  $n \leq 12$ . Let  $\mathcal{S}_{[n]}$  denote the power set on  $n$  elements. For families of sets  $\mathcal{A}$  and  $\mathcal{B}$  define  $\mathcal{A} \uplus \mathcal{B} := \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ . Finally, for a family of sets  $\mathcal{F}$  define  $\mathcal{F}_i := \{F \in \mathcal{F} \mid i \in F\}$ , and denote by  $\mathcal{X}^{\mathcal{F}}$  the incidence vector of  $\mathcal{F}$ .

**Theorem 1 (Poonen 1992).** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements. The following statements are equivalent:*

1. *For every UC family  $\mathcal{F} \supseteq \mathcal{A}$ , there exists  $i \in [n]$  such that  $|\mathcal{F}_i| \geq |\mathcal{F}|/2$ .*
2. *There exist non-negative real numbers  $c_1, \dots, c_n$  with  $\sum_{i \in [n]} c_i = 1$  such that for every UC family  $\mathcal{B} \subseteq \mathcal{S}_{[n]}$  with  $\mathcal{B} \uplus \mathcal{A} = \mathcal{S}_{[n]}$ , the following inequality holds*

$$\sum_{i \in [n]} c_i |\mathcal{B}_i| \geq |\mathcal{B}|/2. \quad (1)$$

The proof of the theorem includes a beautiful application of the separating hyperplane theorem and points, at least algorithmically, in the right way. Indeed, for a fixed UC family  $\mathcal{A}$  that covers  $n$  elements, it is easy to see that the second statement above describes a convex polyhedron  $P_c \subset \mathbb{R}^n$  defined by:

$$\left\{ \begin{array}{l} \sum_{i \in [n]} c_i = 1; \\ \sum_{i \in [n]} c_i |\mathcal{B}_i| \geq |\mathcal{B}|/2 \quad \forall \mathcal{B} \subseteq \mathcal{S}_{[n]} : \mathcal{B} \uplus \mathcal{A} = \mathcal{S}_{[n]}; \\ c_i \geq 0 \quad \forall i \in [n]; \end{array} \right\}$$

Furthermore since the coefficients are all integral, if there exists a feasible point of  $P_c$ , we can safely assume (via Fourier-Motzkin elimination) that the  $c_i$  are rational. Thus, we can use the simplex or interior point methods to find a feasible point of  $P_c$ , or show that one does not exist via Farkas’ Lemma. Furthermore

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<sup>1</sup> Final computations are rechecked with CPLEX 12.6.3 [7], Gurobi 6.5.2 [10], and exact SCIP [6]. For  $n \geq 8$ , we use CPLEX 12.6.3 [7] for the separation oracle, then recheck the results with the rest of the solvers. In addition, the branch and bound tree of exact SCIP [6] is verified with VIPR [5].

let  $P_{\bar{c}}$  denote the following integer program:

$$\begin{aligned}
& \min \sum_{i \in [n]} \bar{c}_i \\
& \text{s.t. } \sum_{i \in [n]} \bar{c}_i |\mathcal{B}_i| \geq (|\mathcal{B}|/2) \sum_{i \in [n]} \bar{c}_i \quad \forall \mathcal{B} \subseteq \mathcal{S}_{[n]} : \mathcal{B} \uplus \mathcal{A} = \mathcal{B} \\
& \sum_{i \in [n]} \bar{c}_i \geq 1 \\
& \bar{c}_i \in \mathbb{Z}_{\geq 0} \quad \forall i \in [n]
\end{aligned}$$

**Observation 2** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements. Then there exists a feasible point of  $P_c$  if and only if there exists a feasible solution of  $P_{\bar{c}}$ .*

*Proof.* We simply scale. More precisely, let  $p_c$  be a feasible point of  $P_c$ . We can safely assume that  $p_c$  is a rational vector, i.e.,  $p_c = \{c_1 = \frac{a_1}{b_1}, c_2 = \frac{a_2}{b_2}, \dots, c_n = \frac{a_n}{b_n}\} \in \mathbb{Q}_{\geq 0}^n$ . Define  $g := \text{lcm}(b_1, b_2, \dots, b_n)$ , and  $\bar{c}_i := gc_i \in \mathbb{Z}_{\geq 0}$  for all  $i \in [n]$ . It is easy to see that the defined  $\bar{c}_i$  yield a feasible solution of  $P_{\bar{c}}$ . Let  $p_{\bar{c}}$  be a feasible solution of  $P_{\bar{c}}$ , i.e.,  $p_{\bar{c}} = \{\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n\} \in \mathbb{Z}_{\geq 0}^n$ . Define  $c_i := \bar{c}_i / (\sum_{i \in [n]} \bar{c}_i)$ . It is easy to see that the defined  $c_i$  yield a feasible point of  $P_c$ .  $\square$

A *separation oracle* for a polyhedron  $P \subset \mathbb{R}^n$  is an algorithm that, queried on  $x \in \mathbb{R}^n$ , either asserts that  $x \in P$  or returns  $h \in \mathbb{R}^n$  such that  $hy < hx$  for all  $y \in P$ . In order to design a separation oracle for  $P_c$  (or  $P_{\bar{c}}$ ), we first need the following corollary of Poonen's Theorem, a version of which is already noted in Morris [13]. We formalize it again here for clarity and reference.

**Corollary 1.** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements. The following are equivalent:*

1. *Frankl's conjecture holds for every UC family  $\mathcal{F} \supseteq \mathcal{A}$ . In particular, there exists  $i \in [n]$  such that  $|\mathcal{F}_i| \geq |\mathcal{F}|/2$ .*
2. *For every  $\mathcal{B} \subseteq \mathcal{S}_n$  with  $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ , there exist  $c_i \in \mathbb{Q}_{\geq 0}$  for all  $i \in [n]$ , with  $\sum_{i \in [n]} c_i = 1$ , such that  $\sum_{S \in \mathcal{B}} (\sum_{i \in S} c_i - \sum_{i \notin S} c_i) \geq 0$ .*

*Proof.* Fix an UC family  $\mathcal{B} \subseteq S_n$  with  $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ . Then the following holds,

$$\begin{aligned}
\sum_{S \in \mathcal{B}} \left( \sum_{i \in S} c_i - \sum_{i \notin S} c_i \right) &= 2 \sum_{S \in \mathcal{B}} \sum_{i \in S} c_i - \sum_{S \in \mathcal{B}} \left( \sum_{i \notin S} c_i + \sum_{i \in S} c_i \right) \\
&= 2 \sum_{S \in \mathcal{B}} \sum_{i \in S} c_i - \sum_{S \in \mathcal{B}} \sum_{i \in [n]} c_i \\
&= 2 \sum_{i \in [n]} c_i |\mathcal{B}_i| - |\mathcal{B}| \sum_{i \in [n]} c_i \geq 0 \\
&\iff \sum_{i \in [n]} c_i |\mathcal{B}_i| \geq |\mathcal{B}|/2.
\end{aligned}$$

Since the above holds for every UC family  $\mathcal{B} \subseteq S_n$  with  $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ , the desired result follows from Poonen's Theorem.  $\square$

Poonen [14] shows that in order to prove that Frankl's conjecture holds for UC families of all cardinalities it is sufficient to consider UC families that contain the empty set. In the next proposition, we show that for FC or Non-FC-families we can always assume (when convenient) that the empty set is present.

**Proposition 1.** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements such that  $\emptyset \in \mathcal{A}$ . Then  $\mathcal{A}$  is a FC-family if and only if  $\mathcal{A} \setminus \emptyset$  is a FC-family.*

*Proof.* Let  $\mathcal{A}$  be an UC family that covers  $n$  elements such that  $\emptyset \in \mathcal{A}$ . Suppose  $\mathcal{A}$  is a FC-family. Then for all UC families  $\mathcal{F} \supseteq \mathcal{A}$  there exists  $i \in [n]$  such that  $|\mathcal{F}_i| \geq |\mathcal{F}|/2$ . Hence  $\mathcal{F} \setminus \emptyset$  also satisfies Frankl's conjecture. Therefore  $\mathcal{A} \setminus \emptyset$  is a FC-family. Suppose  $\mathcal{A} \setminus \emptyset$  is a FC-family. As before for all UC families  $\mathcal{F} \supseteq \{\mathcal{A} \setminus \emptyset\}$  Frankl's conjecture is satisfied. Let  $\mathcal{T}_{\mathcal{F}}$  be the collection of all such families. Then  $\mathcal{T}_{\mathcal{F}} = \mathcal{T}_{\tilde{\mathcal{F}}} \cup \mathcal{T}_{\overline{\mathcal{F}}}$ , where  $\mathcal{T}_{\tilde{\mathcal{F}}} := \left\{ \tilde{\mathcal{F}} \supseteq \{\mathcal{A} \setminus \emptyset\} \mid \emptyset \notin \tilde{\mathcal{F}} \right\}$  and  $\mathcal{T}_{\overline{\mathcal{F}}} := \left\{ \overline{\mathcal{F}} \supseteq \{\mathcal{A} \setminus \emptyset\} \mid \emptyset \in \overline{\mathcal{F}} \right\}$ . Define  $\mathcal{T}_{\mathcal{D}} := \{\mathcal{D} \supseteq \mathcal{A} \mid \mathcal{D} \text{ is an UC family}\}$ . Then it is clear that  $\mathcal{T}_{\mathcal{F}} = \mathcal{T}_{\mathcal{D}}$ , which implies that  $\mathcal{A}$  is a FC-family.  $\square$

Corollary 1 combined with an integer programming approach to UC families inspired by the work of Pulaj, Raymond and Theis [15], provides the basis of our separation oracle. Fix an UC family  $\mathcal{A} \subseteq S_{[n]}$  that covers  $n$  elements such that  $\emptyset \in \mathcal{A}$ , and fix  $\mathcal{C} := \{c_1 = \frac{a_1}{b_1}, c_2 = \frac{a_2}{b_2}, \dots, c_n = \frac{a_n}{b_n}\} \in \mathbb{Q}_{\geq 0}^n$  with  $\sum_{i \in [n]} c_i = 1$ . Furthermore, define  $g := \text{lcm}(b_1, b_2, \dots, b_n)$ , and  $\bar{c}_i := gc_i \in \mathbb{Z}_{\geq 0}$  for all  $i \in [n]$  as previously. Let  $FC(\mathcal{A}, \mathcal{C})_n$  denote the following polyhedron:

$$\left\{ \begin{array}{ll} x_T + x_U \leq 1 + x_S & \forall T \cup U = S \in \mathcal{S}_{[n]}; \\ \sum_{S \in \mathcal{S}_n} (\sum_{i \in S} \bar{c}_i - \sum_{i \notin S} \bar{c}_i) x_S + 1 \leq 0; & \\ x_T \leq x_U & \forall S \cup T = U \in \mathcal{S}_{[n]} : S \in \mathcal{A}; \\ x_S \in \{0, 1\} & \forall S \in \mathcal{S}_{[n]}; \end{array} \right\}$$

Suppose  $FC(\mathcal{A}, \mathcal{C})_n$  is non-empty, and let  $p^* \in FC(\mathcal{A}, \mathcal{C})_n$ . Then  $p^* = \mathcal{X}^{\mathcal{B}}$  where  $\mathcal{B} \subseteq \mathcal{S}_{[n]}$ . The first inequalities ensure that the chosen family  $\mathcal{B}$  is UC, and we denote them as UC inequalities. The third class of inequalities ensures that  $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ , and we denote them as Fixed-Set (FS) inequalities.<sup>2</sup> We explain the  $\bar{c}_i$  inequality in the next proposition.

**Proposition 2.** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements, and let  $\mathcal{C} \in \mathbb{Q}_{\geq 0}^n$  with  $\sum_{i \in [n]} c_i = 1$ . If  $FC(\mathcal{A}, \mathcal{C})_n$  is unfeasible, then all UC families  $\mathcal{F} \supseteq \mathcal{A}$  satisfy Frankl's conjecture.*

*Proof.* Suppose that  $FC(\mathcal{A}, \mathcal{C})_n$  is unfeasible. Let  $\tilde{P}$  be defined as  $FC(\mathcal{A}, \mathcal{C})_n$  without the  $\bar{c}_i$  inequality. It is easy to see that  $\tilde{P}$  is non-empty since the all zero vector is feasible. Suppose  $\mathcal{B} \subseteq \mathcal{S}_n$  is an UC family. Then any  $p^* = \mathcal{X}^{\mathcal{B}}$  with  $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$  is feasible. Therefore if  $FC(\mathcal{A}, \mathcal{C})_n$  is unfeasible this implies there exists no UC family  $\mathcal{B} \subseteq \mathcal{S}_n$  with  $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$  such that:

$$\sum_{S \in \mathcal{B}} \left( \sum_{i \in S} \bar{c}_i - \sum_{i \notin S} \bar{c}_i \right) \leq -1.$$

Since  $x_S = \{0, 1\}$  for all  $S \in \mathcal{S}_{[n]}$ , and  $\bar{c}_i \in \mathbb{Z}_{\geq 0}$  for all  $i \in [n]$ , this implies that for all UC families  $\mathcal{B} \subseteq \mathcal{S}_n$  with  $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ , the following inequality holds:

$$\sum_{S \in \mathcal{B}} \left( \sum_{i \in S} \bar{c}_i - \sum_{i \notin S} \bar{c}_i \right) \geq 0.$$

It is easy to see that corollary 1 still holds if we replace  $c_i$  with  $\bar{c}_i$ . Therefore Poonen's Theorem implies that all UC families  $\mathcal{F} \supseteq \mathcal{A}$  satisfy the Frankl conjecture.  $\square$

A natural candidate for checking the feasibility of  $FC(\mathcal{A}, \mathcal{C})_n$ , for some  $\mathcal{A}$  and  $\mathcal{C}$ , is a standard branch and bound algorithm. Since we are mainly interested in proving that certain (previously unknown) UC families  $\mathcal{A}$  are FC or Non-FC, we do not address questions of complexity, but simply optimize some linear objective function over  $FC(\mathcal{A}, \mathcal{C})_n$  in a general purpose integer programming solver as specified in section 1. We do the same for optimizing over  $P_{\bar{c}}$ , and checking the feasibility of  $P_{\bar{c}}$ . For the remainder of the paper  $\mathcal{C}$  can refer either to a vector of non-negative rationals that equal one as above, or (equivalently via scaling) to a vector of non-negative integers which sum up to at least one.

<sup>2</sup> If we drop the assumption that  $\emptyset \in \mathcal{A}$  we arrive at  $\mathcal{B} \uplus \mathcal{A} \subseteq \mathcal{B}$ , an equivalent condition we find in Vaughan [17], [18], [19].

**Corollary 2.** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements. Given  $\mathcal{C} \in \mathbb{Q}_{\geq 0}^n$  with  $\sum_{i \in [n]} c_i = 1$  as input, determining whether there exists a feasible point  $\bar{p}^*$  of  $FC(\mathcal{A}, \mathcal{C})_n$  is equivalent to a separation oracle for  $P_c$ .*

*Proof.* Suppose  $p^* = \mathcal{X}^{\mathcal{B}}$  is a feasible point of  $FC(\mathcal{A}, \mathcal{C})_n$ . As a result, we arrive at the following valid inequality for  $P_c$  (where we distinguish the variables  $\tilde{c}_i \geq 0$  from the fixed  $c_i$ , for all  $1 \leq i \leq n$ )

$$\sum_{i \in [n]} \tilde{c}_i |\mathcal{B}_i| \geq |\mathcal{B}|/2,$$

which separates  $\mathcal{C}$  from  $P_c$  since the following implications hold

$$\sum_{S \in \mathcal{B}} \left( \sum_{i \in S} \tilde{c}_i - \sum_{i \notin S} \tilde{c}_i \right) \leq -1 \iff |\mathcal{B}|/2 - \sum_{i \in [n]} c_i |\mathcal{B}_i| > 0.$$

Otherwise, if  $FC(\mathcal{A}, \mathcal{C})_n$  is unfeasible, no such valid inequality exists. In this case, proposition 2 implies that  $\mathcal{C} \in P_c$ .  $\square$

Furthermore, define as  $FC(\mathcal{A}, \mathcal{C})_n^{Max}$  the binary program which maximizes the following linear objective function

$$\sum_{i \in [n]} \tilde{c}_i \left( \sum_{S \in \mathcal{S}_{[n]}} x_S - 2 \sum_{S \in \mathcal{S}_{[n]}; i \in S} x_S \right),$$

over  $FC(\mathcal{A}, \mathcal{C})_n$ .

**Observation 3** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements, and let  $\mathcal{C} \in \mathbb{Q}_{\geq 0}^n$  with  $\sum_{i \in [n]} c_i = 1$ . An optimal solution of  $FC(\mathcal{A}, \mathcal{C})_n^{Max}$  returns a maximally violated inequality for  $P_c$ .*

It is easy to see that in the following algorithm the tuples  $(P_c, FC(\mathcal{A}, \mathcal{C})_n^{Max})$  and  $(P_c, FC(\mathcal{A}, \mathcal{C})_n)$  may be used interchangeably with appropriate adjustments.



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**Algorithm 1:** Cutting planes for FC-families

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**Data:** An UC family  $\mathcal{A}$  that covers  $n$  elements.

**Result:**  $c_i$  for  $\mathcal{A}$ , or infeasible  $P_k \supseteq P_c$ .

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1  $P_k \leftarrow \left( \sum_{i \in [n]} c_i = 1, c_i \geq 0, \forall i \in [n] \right), \mathcal{C} \leftarrow \emptyset, g \leftarrow \emptyset;$ 
2 while  $\exists C \in P_k$  do
3    $g \leftarrow \text{lcm}(b_1, b_2, \dots, b_n)$ , where
    $C = \{c_1 = \frac{a_1}{b_1}, c_2 = \frac{a_2}{b_2}, \dots, c_n = \frac{a_n}{b_n}\} \in \mathbb{Q}_{\geq 0}^n$ ;
4    $\mathcal{C} \leftarrow \{gc_1, gc_2, \dots, gc_n\}$ ;
5   if  $\exists p^* \in FC(\mathcal{A}, \mathcal{C})_n$  then
6      $P_k \leftarrow P_k \cap \left( \sum_{i \in [n]} c_i |\mathcal{B}_i| \geq |\mathcal{B}|/2 \right)$ , where  $p^* = \mathcal{X}^{\mathcal{B}}$ ;
7   else
8     return  $C$ 
9 return  $P_k \supseteq P_c$ 
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**Theorem 4.** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements. Then Algorithm 1 either outputs weights which prove that  $\mathcal{A}$  is FC, or an infeasible system of constraints which proves that  $\mathcal{A}$  is Non-FC.*

*Proof.* It is clear algorithm 1 finitely terminates. Let  $\mathcal{A}$  be an UC family that covers  $n$  elements. Suppose  $\mathcal{A}$  is a FC-family. By Poonen's Theorem there exist  $c_i \geq 0$  for all  $i \in [n]$  with  $\sum_{i \in [n]} c_i = 1$  that satisfy all inequalities of type 1. Given  $\mathcal{A}$ , at some iteration of algorithm 1, for some  $\mathcal{C}$ , by corollary 2 we arrive at  $C \in P_c$ , otherwise if the algorithm terminates without outputting some  $C \in P_c$ , it outputs an infeasible  $P_k \supseteq P_c$  which implies that  $\mathcal{A}$  is not a FC-family and we arrive at a contradiction. Suppose  $\mathcal{A}$  is a Non-FC-family. This implies there exist no  $c_i \geq 0$  for all  $i \in [n]$  with  $\sum_{i \in [n]} c_i = 1$  that satisfy all inequalities of type 1. By corollary 2 during all the iterations of algorithm 1 we have that  $C \notin P_c$ , otherwise we arrive at a contradiction. Therefore algorithm 1 terminates when  $P_k = \emptyset$ , which implies that  $P_k \supseteq P_c$  is infeasible.  $\square$

### 3 Valid Inequalities

From the perspective of computational integer programming, valid inequalities are considered effective if they lead to a smaller branch and bound tree. For all the results that we feature in this paper, adding a subset of the following inequalities to the root node significantly reduces the size of the branch and bound tree. For small  $n$  this allows for manual inspection, as we illustrate in the appendix. A family of sets  $\mathcal{S}$  generates  $\mathcal{F}$  with  $\mathcal{A}$ , denoted by  $\langle \mathcal{S} \rangle_{\mathcal{A}} := \mathcal{F}$ , if and only if  $\mathcal{F}$  is the smallest UC family that contains  $\mathcal{S}$  such that  $\mathcal{F} \uplus \mathcal{A} = \mathcal{F}$ .

**Proposition 3 (FC inequalities).** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements, and let  $\mathcal{C} \in \mathbb{Q}_{\geq 0}^n$  with  $\sum_{i \in [n]} c_i = 1$ . Suppose we have  $S \in \mathcal{A}$ ,  $S \cup U = F$ ,*

and  $S \cup T = F$ . Then the following inequality

$$x_T + x_U - x_{T \cup U} - x_F \leq 0,$$

is valid for  $FC(\mathcal{A}, \mathcal{C})_n$ .

*Proof.* Suppose there exists a feasible solution of  $FC(\mathcal{A}, \mathcal{C})_n$  which yields an UC family  $\mathcal{F}$  such that the following inequality holds

$$x_T + x_U - x_{T \cup U} - x_F \geq 1.$$

This implies that the number of variables which equal one with positive coefficients is greater than the number of variables with negative coefficients which equal one. But if either  $x_T$  or  $x_U$  are one then  $x_F$  is one (if both are one then  $x_{T \cup U}$  is one) and we arrive at a contradiction.  $\square$

In the following definition the role of a considered UC family  $\mathcal{A}$  is taken into account in the listed conditions. In the first condition the role of  $\mathcal{A}$  is implicit in the existence of a FS inequality, whereas in the second condition the role of  $\mathcal{A}$  is implicit in *generating* the desired family, as discussed at the beginning of this section.

**Definition 1 (FC-chain).** Let  $\mathcal{A}$  be an UC family that covers  $n$  elements, and let  $\mathcal{C} \in \mathbb{Q}_{\geq 0}^n$  with  $\sum_{i \in [n]} c_i = 1$ . Let  $\mathcal{S}, \mathcal{S}' \subset \mathcal{S}_{[n]}$ ,  $\mathcal{S} \cap \mathcal{S}' = \emptyset$ . Given  $B_i \in \mathcal{S}, B_j \in \mathcal{S}'$ , we say  $B_i, B_j$  form a FC-chain which we denote by  $B_i \longrightarrow B_j$ , if there exist tuples  $(B_i, B_1), (B_1, B_2), \dots, (B_m, B_j)$ ,  $B_k \in \mathcal{S}_{[n]}$  for all  $1 \leq k \leq m$ , such that for any tuple  $(B_i, B_p)$ , at least one of the following conditions holds:

1. There exists  $A \in \mathcal{A}$  such that  $x_{B_i} \leq x_{B_p}$  is a valid FS inequality for  $FC(\mathcal{A}, \mathcal{C})_n$ .
2. There exists  $S \in \langle \mathcal{S} \rangle_{\mathcal{A}}$  such that  $x_{B_i} + x_S \leq 1 + x_{B_p}$  is a valid UC inequality for  $FC(\mathcal{A}, \mathcal{C})_n$ .

We denote an explicit FC-chain by  $B_i \xrightarrow{S} B_1 \xrightarrow{U} \dots B_m \xrightarrow{T} B_j$ , where  $S, U, T$  satisfy either condition listed above. When needed we specify which type of inequalities form a FC-chain by  $S^{UC}, U^{UC}, T^{UC}$  for UC inequalities, and  $S^{FS}, U^{FS}, T^{FS}$  for FS inequalities.

**Proposition 4 (FC-chain inequalities).** Let  $\mathcal{A}$  be an UC family that covers  $n$  elements, and let  $\mathcal{C} \in \mathbb{Q}_{\geq 0}^n$  with  $\sum_{i \in [n]} c_i = 1$ . Let  $\mathcal{S}, \mathcal{S}' \subset \mathcal{S}_{[n]}$ ,  $\mathcal{S} \cap \mathcal{S}' = \emptyset$ . For any  $\mathcal{T} \subseteq \mathcal{S}$  define  $\mathcal{U}_{\mathcal{T}} := \{S' \in \mathcal{S}' \mid \exists S \in \mathcal{T} : S \longrightarrow S'\}$ . Suppose that  $|\mathcal{T}| \leq |\mathcal{U}_{\mathcal{T}}|$  for all  $\mathcal{T} \subseteq \mathcal{S}$ . Then the following inequality

$$\sum_{S \in \mathcal{S}} x_S - \sum_{S \in \mathcal{S}'} x_S \leq 0,$$

is valid for  $FC(\mathcal{A}, \mathcal{C})_n$ .

*Proof.* Suppose there exists a feasible solution of  $FC(\mathcal{A}, \mathcal{C})_n$  which yields an UC family  $\mathcal{F}$  such that the following inequality holds

$$\sum_{S \in \mathcal{S} \cap \mathcal{F}} x_S - \sum_{S \in \mathcal{S}' \cap \mathcal{F}} x_S \geq 1.$$

It is clear that  $\mathcal{S} \cap \mathcal{F} \neq \emptyset$ , otherwise we arrive at a contradiction. Therefore the inequality implies that the number of variables  $x_S$  which equal one, for all  $S \in \mathcal{S} \cap \mathcal{F}$  is greater than the number of variables  $x_S$  which equal one, for all  $S \in \mathcal{S}' \cap \mathcal{F}$ . Let  $\mathcal{T} \subseteq \mathcal{S} \cap \mathcal{F}$ , and for all  $S \in \mathcal{T}$ , let  $x_S = 1$ . Then  $|\mathcal{T}| \leq |\mathcal{U}_T|$  holds by hypothesis. Furthermore by the definition of a FC-chain, for all  $S' \in \mathcal{U}_T$  we conclude that  $x_{S'} = 1$ . Thus we arrive at a contradiction.  $\square$

## 4 Generators for Non-FC-families

In this section we exhibit a counterexample to a conjecture of Morris [13] about generators for Non-FC-families. A generator  $\mathcal{S}$  for a UC family  $\mathcal{F}$  is *minimal* if and only if there exists no  $\mathcal{S}' \subset \mathcal{S}$  such that  $\mathcal{S}'$  is a generator for  $\mathcal{F}$ . In the following definition, it is sufficient to consider  $i \in [n + 1]$ .

**Definition 2 (regular).** *Let  $\mathcal{S}$  be a family of sets that covers  $n$  elements. Suppose  $\mathcal{S}$  is a minimal generator for an UC family  $\mathcal{F}$ , such that  $\mathcal{F}$  is a Non-FC-family. Then  $\mathcal{S}$  is regular if and only if for any  $A \in \mathcal{S}$ ,  $A \neq \emptyset$ , the UC family  $\langle (\mathcal{S} \setminus \{A\}) \cup \{A \cup \{i\}\} \rangle$  is Non-FC.*

*Conjecture 1 (Morris 2006).*  $\mathcal{S}$  is regular for all  $n \in \mathbb{Z}_{\geq 3}$ .<sup>3</sup>

Morris [13] checked the conjecture for all known families at the time, and therefore considered it plausible. In some sense, conjecture 1 perfectly illustrates our general lack of knowledge about UC families since – as a number of other related questions – it has eluded an answer for a relatively long time. The obstacle – in this case and others to follow – is the lack of a method for exactly characterizing FC-families, a gap in knowledge which we correct with our framework. This highlights the importance of appropriate computational techniques in areas where such applications are rarely encountered, but are otherwise quite effective tools. Our counterexample on six elements is minimal, in the sense that Morris [13] completely characterizes FC-families on 5 elements. Let  $\mathcal{S} = \{\emptyset, \{4, 5, 6\}, \{1, 3, 4\}, \{1, 2, 5, 6\}, \{1, 2, 3, 4\}\}$ . Furthermore, let  $\mathcal{T} = \{\{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}$ . Hence it follows that  $\langle \mathcal{S} \rangle = \mathcal{S} \cup \mathcal{T}$ . It is easy to see that  $\mathcal{S}$  is a minimal generator for  $\mathcal{S} \cup \mathcal{T}$ . We will show that  $\langle \mathcal{S} \rangle$  is a Non-FC-family. There is a stronger connection between the structure of inequalities featured in the proof below and questions of Vaughan [18] and Morris [13] we answer later in this work. In section 5 we explicitly describe the structure of UC families from which the inequalities below are derived in relation to the questions of interest.

<sup>3</sup> It is well-known and easy to show that 1-sets and 2-sets (as done in Sarvate and Renaud [16]) are FC-families.

**Proposition 5.**  $\mathcal{S} \cup \mathcal{T}$  is a Non-FC-family.

*Proof.* The proof is the output of algorithm 1 with  $S \cup \mathcal{T}$  as input, which is an infeasible system of constraints. We display an irreducible infeasible subset of the given system. We identify columns with zero one entries for each  $S \in \mathcal{S}_6$ . The six matrices featured below represent UC families. The top row keeps track of the number of sets in each family. In addition to rechecking with an exact rational solver [6] and other solvers, we check that each matrix is UC via simple external subroutines and finally by hand.<sup>4</sup> Furthermore, let  $\mathcal{F}$  be a family represented by one of the matrices below. It easy to see that  $\mathcal{F} \sqcup \langle \mathcal{S} \rangle = \mathcal{F}$  by inspection. In each matrix, we color columns which correspond to sets in  $\mathcal{S}$ ,  $\mathcal{T}$ , red and blue, respectively. Each matrix yields an inequality of type 1 (multiplied by two) featured below it. The following system of constraints is infeasible in non-negative  $c_i$  for all  $1 \leq i \leq 6$ . For each row we display the Farkas dual values in square brackets. This yields a certificate of infeasibility via a straightforward application of Farkas' Lemma. For convenience we state the lemma in the appendix.

$$[-7190] : c_1 + c_2 + c_3 + c_4 + c_5 + c_6 = 1.$$

[illegible]

$$[30] : 22c_1 + 46c_2 + 50c_3 + 50c_4 + 46c_5 + 46c_6 \geq 43.$$

$c_1$	1	2	3	4	5	6	7	8	9	0	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	
$c_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$c_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
$c_4$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1
$c_5$	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	0	0	0	0	1	1	1	1	1	0	0	0	1	1	1	1	1	0	1	0	1	1	1	1
$c_6$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	1	1	0	0	1	0	0	1	1	1	1	1	0	0	1	1
$c_7$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	1	0	1	0

$$[9] : 46c_1 + 14c_2 + 42c_3 + 42c_4 + 42c_5 + 42c_6 \geq 39.$$

[illegible]

$$[44]: 52c_1 + 46c_2 + 52c_3 + 28c_4 + 52c_5 + 52c_6 \geq 46.$$

<sup>4</sup> The reader does not need to rely on these tedious checks to ensure the correctness of the result as we show in section 5.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40			
$c_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
$c_2$	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
$c_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1
$c_4$	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1
$c_5$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
$c_6$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	

$$[21] : 48c_1 + 40c_2 + 16c_3 + 48c_4 + 40c_5 + 40c_6 \geq 40.$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
$c_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$c_2$	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
$c_3$	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_4$	0	0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_5$	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_6$	0	1	0	1	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

$$[32] : 44c_1 + 44c_2 + 42c_3 + 48c_4 + 20c_5 + 52c_6 \geq 42.$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
$c_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$c_2$	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_3$	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_4$	0	0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_5$	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_6$	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$$[32] : 44c_1 + 44c_2 + 42c_3 + 48c_4 + 52c_5 + 20c_6 \geq 42.$$

□

**Proposition 6.** Let  $S' = \{\emptyset, \{4, 5, 6\}, \{1, 3, 4\}, \{1, 2, 5, 6\}, \{1, 2, 3, 4, 5\}\}$ . Then  $\langle S' \rangle$  is a FC-family.

*Proof.* Let  $\mathcal{C} = \{16, 8, 12, 20, 17, 15\}$ . Then  $FC(\langle S' \rangle, \mathcal{C})_n$  is infeasible<sup>5</sup>. □

**Corollary 3.**  $\mathcal{S}$  is a counterexample to conjecture 1.

*Proof.*  $S' = (\mathcal{S} \setminus \{1, 2, 3, 4\}) \cup \{\{1, 2, 3, 4\} \cup \{5\}\}$ . □

## 5 The Relaxation Questions

In this section, we briefly address the practical behaviour of our algorithm – with a focus on the number of calls made to our separation oracle – as it sheds light on open questions of interests in Vaughan [18] and Morris [13]. As a result, we exhibit a counterexample to both closely related questions.

<sup>5</sup> In the appendix we explicitly show the infeasibility of  $FC(S', \mathcal{C})_n$  by making use of FC-chain inequalities and displaying irreducible infeasible subsets of constraints for the two leaf nodes of the resulting branch and bound tree.

Although throughout this work we mostly refer to  $(P_c, FC(\mathcal{A}, \mathcal{C})_n)$  to keep the exposition clear, our current implementation features  $(P_{\bar{c}}, FC(\mathcal{A}, \mathcal{C})_n^{Max})$ . In doing so we avoid possible numerical trouble by minimizing the sum of the  $c_i$ , in addition to selecting the “sharpest cut” whenever the separation oracle is called. Yet, without witnessing first-hand computations for fixed UC families  $\mathcal{A}$  that covers  $6 \leq n \leq 10$ , our separation oracle and algorithm appear fraught with theoretical dangers.<sup>6</sup> However, in practise our method is well-behaved in the described range, and is consequently the currently best available technique for the *exact* determination of FC-families.

Furthermore, our implementation *mostly* confirms the heuristic intuition of Vaughan and Morris as will be made explicit in the next paragraphs. Thus in the tested range, our separation oracle is mostly called  $n$  times. However, in some cases the oracle is called more than  $n$  (but less than  $2n$ ) times<sup>7</sup>. Among the later we find counterexamples to open questions of interest which we feature below.

As mentioned in section 1, Vaughan [18] implements a heuristic that guides the search for a feasible weight system. Given an UC family  $\mathcal{A}$ ,  $\emptyset \in \mathcal{A}$ , the heuristic focuses only on UC families  $\mathcal{B}$  with  $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ , where  $\mathcal{B} = S_{[n] \setminus \{j\}} \uplus \mathcal{A}$  for all  $j \in [n]$ . If there exists a solution to the system of linear equations  $\sum_{i \in [n]} c_i |\mathcal{B}_i| = |\mathcal{B}|/2$  in non-negative  $c_i$ , with  $\sum_{i \in [n]} c_i \leq 1$ , then the considered UC family  $\mathcal{A}$  becomes a candidate FC-family. All of Vaughan’s candidate FC-families in [18] are identified as above, followed by tedious case analysis that spans several pages for the proof that the given family is FC. We precisely state Vaughan’s question as follows:

*Question 1 (Vaughan 2002).* Let  $\mathcal{A}$ ,  $\emptyset \in \mathcal{A}$  be an UC family that covers  $n$  elements, and consider UC families  $\mathcal{B} = S_{[n] \setminus \{j\}} \uplus \mathcal{A}$  for all  $j \in [n]$ . Suppose the linear system of equations  $\sum_{i \in [n]} c_i |\mathcal{B}_i| = |\mathcal{B}|/2$  for all  $\mathcal{B}$  as above has a solution in non-negative  $c_i$  for all  $i \in [n]$ , such that  $\sum_{i \in [n]} c_i \leq 1$ . Denote this by  $NUM(\mathcal{A}) \leq 1$ . Does  $NUM(\mathcal{A}) \leq 1$  imply that  $P_c$  is feasible?

Given an UC family  $\mathcal{A}$ ,  $\emptyset \in \mathcal{A}$ , Morris [13] also focuses on  $\mathcal{B}$  as above, searching instead for feasible non-negative points of the polytope defined by the  $n$  given  $\mathcal{B}$  and  $\sum_{i \in [n]} c_i \geq 1$  when the desired points are integral, or  $\sum_{i \in [n]} c_i = 1$  when the desired points are rational. The idea is that the  $n$  given inequalities could serve as a relaxation of the underlying polytope of  $P_{\bar{c}}$  (or equivalently  $P_c$ ). Morris shows that this holds in a number of cases, but is it true in general? More precisely, we state it as the following question:

*Question 2 (Morris 2006).* Let  $\mathcal{A}$ ,  $\emptyset \in \mathcal{A}$  be an UC family that covers  $n$  elements, and consider UC families  $\mathcal{B} = S_{[n] \setminus \{j\}} \uplus \mathcal{A}$  for all  $j \in [n]$ . Suppose the polytope

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<sup>6</sup> The separation oracle is a binary program with an exponential number of variables and constraints in  $n$ , and furthermore the number of calls it receives could be exponential.

<sup>7</sup> The runtimes vary roughly from a few seconds for  $6 \leq n \leq 7$  and a few minutes for  $8 \leq n \leq 9$ , to a few hours for  $n = 10$ . Furthermore verification with exact SCIP [6] takes longer, as does testing a non-minimal FC-family. Computations were carried out on machines with 2.40 GHz quad-core processors and 16 GB of RAM.

$\widetilde{P}_{\bar{c}}(\mathcal{A})$  defined by  $\sum_{i \in [n]} \bar{c}_i \geq 1$ ,  $\sum_{S \in \mathcal{B}} (\sum_{i \in S} \bar{c}_i - \sum_{i \notin S} \bar{c}_i) \geq 0$  for all  $\mathcal{B}$  as above, and  $\bar{c}_i \in \mathbb{Z}_{\geq 0}$  for all  $i \in [n]$ , is non-empty. Does this imply  $P_{\bar{c}}$  is feasible?

**Observation 5** *A positive answer to Vaughan's question implies a positive answer to Morris' question.*

Thus, considering the above, we can explicitly describe the structure associated with the Non-FC-family that leads to the counterexample in section 1. As above, it suffices to consider  $\mathcal{B} = S_{[n] \setminus \{j\}} \uplus \mathcal{A}$  for all  $j \in [n]$ , where  $\mathcal{A}$  is our given UC family. This greatly simplifies the tedious task of checking that the algorithm's output is correct. Once the family is constructed according to the given  $\mathcal{B}$ , it is easy to see that the necessary conditions for correctness are met. Given that the empty set does not make a difference in determining whether an UC family  $\mathcal{A}$  is FC or Non-FC, as we saw in proposition 1, we may think the condition  $\emptyset \in \mathcal{A}$  in the questions of Vaughan and Morris can be relaxed. If this were the case, the structure of the considered  $\mathcal{B}$  with  $\emptyset \notin \mathcal{A}$  is again simplified, since the cardinality of the new family is at most the cardinality of the original one. Unfortunately, as we shall see, this is not the case. Still, in the next proposition, we show that a feasible  $\widetilde{P}_{\bar{c}}(\mathcal{A})$  implies the feasibility of a polytope arising from smaller structures.

**Proposition 7.** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements such that  $\emptyset \in \mathcal{A}$ . Suppose  $\widetilde{P}_{\bar{c}}(\mathcal{A})$  is non-empty. Then the analogous polytope defined by the constraints derived from  $\{S_{[n] \setminus \{j\}} \uplus \mathcal{A}\} \setminus S_{[n] \setminus \{j\}}$ , for all  $j \in [n]$ , is non-empty.*

*Proof.* Let  $\mathcal{A}$  be an UC family that covers  $n$  elements such that  $\emptyset \in \mathcal{A}$ . Furthermore, let  $\mathcal{B} = S_{[n] \setminus \{1\}} \uplus \mathcal{A}$ . Since  $\emptyset \in \mathcal{A}$ , it follows that  $S_{[n] \setminus \{1\}} \subset \mathcal{B}$ . Define  $\mathcal{D} := S_{[n] \setminus \{1\}}$ ,  $\tilde{\mathcal{A}} := \mathcal{B} \setminus \mathcal{D}$ . Suppose that  $\widetilde{P}_{\bar{c}}(\mathcal{A})$  is non-empty. From observation 2 we focus our arguments on the rational equivalent of  $\widetilde{P}_{\bar{c}}(\mathcal{A})$  from Poonen's Theorem. Then the following holds,

$$\begin{aligned} \sum_{i \in [n]} 2c_i |\mathcal{B}_i| \geq |\mathcal{B}| &\iff \sum_{i \in [n]} 2c_i |\tilde{\mathcal{A}}_i| + \sum_{i \in [n] \setminus \{1\}} 2c_i |\mathcal{D}_i| \geq |\tilde{\mathcal{A}}| + |\mathcal{D}| \\ &\implies \sum_{i \in [n]} 2c_i |\tilde{\mathcal{A}}_i| \geq |\tilde{\mathcal{A}}|. \end{aligned}$$

In the last implication we use  $\sum_{i \in [n] \setminus \{1\}} c_i \leq 1$ , with  $c_i \geq 0$  for all  $i \in [n] \setminus \{1\}$  and therefore

$$\sum_{i \in [n] \setminus \{1\}} 2c_i |\mathcal{D}_i| = |\mathcal{D}| \sum_{i \in [n] \setminus \{1\}} c_i \leq |\mathcal{D}|.$$

Since the same argument applies to  $\mathcal{B} = S_{[n] \setminus \{j\}} \uplus \mathcal{A}$  for all  $j \in [n]$ , the desired result follows.  $\square$

As we shall see next, the implication does not hold the other way around. Let  $\mathcal{S} = \{\emptyset, \{1, 2, 3\}, \{1, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}\}$ . Let  $\tilde{\mathcal{S}} := \mathcal{S} \setminus \emptyset$ . Morris [13] proves that  $\widetilde{P_c}(\langle \mathcal{S} \rangle)$  is empty.

**Proposition 8.** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements. A feasible  $\widetilde{P_c}(\tilde{\mathcal{A}})$  does not imply a feasible  $\widetilde{P_c}(\mathcal{A})$ .*

*Proof.* We show that  $\widetilde{P_c}(\langle \tilde{\mathcal{S}} \rangle)$  is non-empty. It is easy to see this without explicitly writing down the inequalities. Observe that if we write each set in  $\langle \tilde{\mathcal{S}} \rangle$  as a column of a  $n \times m$  binary matrix  $M$ , we have more entries with ones than zeros. We conclude similarly for all  $j \in [n]$  such that  $\mathcal{B} = S_{[n] \setminus \{j\}} \uplus \mathcal{A}$ . Hence, any vector  $\bar{c}_i \in \mathbb{Z}_{\geq 0}$  for all  $i \in [n]$ , with  $\sum_{i \in [n]} \bar{c}_i \geq 1$ , is a feasible point of  $\widetilde{P_c}(\langle \tilde{\mathcal{S}} \rangle)$ .  $\square$

**Corollary 4.** *The reverse implication in proposition 7 does not hold.*

Finally, we give a negative answer to Morris' question, which in turn implies a negative answer to Vaughan's question.

Let  $\mathcal{S} = \{\emptyset, \{2, 3, 4, 6, 7\}, \{1, 2, 3, 4\}, \{1, 3, 4, 6\}, \{5, 6, 7\}, \{3, 4, 7\}\}$ .

**Proposition 9.**  *$\widetilde{P_c}(\langle \mathcal{S} \rangle)$  is feasible.*

*Proof.* We simply write down the relevant inequalities and exhibit a feasible point. The order of display matches  $j$  in  $\mathcal{B} = S_{[n] \setminus \{j\}} \uplus \langle \mathcal{S} \rangle$ .

$$\begin{aligned} -52\bar{c}_1 + 4\bar{c}_2 + 12\bar{c}_3 + 12\bar{c}_4 + 4\bar{c}_6 &\geq 0 \\ +6\bar{c}_1 - 54\bar{c}_2 + 10\bar{c}_3 + 10\bar{c}_4 + 2\bar{c}_6 + 2\bar{c}_7 &\geq 0 \\ +6\bar{c}_1 + 2\bar{c}_2 - 42\bar{c}_3 + 22\bar{c}_4 + 2\bar{c}_6 + 10\bar{c}_7 &\geq 0 \\ +6\bar{c}_1 + 2\bar{c}_2 + 22\bar{c}_3 - 42\bar{c}_4 + 2\bar{c}_6 + 10\bar{c}_7 &\geq 0 \\ -48\bar{c}_5 + 16\bar{c}_6 + 16\bar{c}_7 &\geq 0 \\ +5\bar{c}_1 + 1\bar{c}_2 + 7\bar{c}_3 + 7\bar{c}_4 + 13\bar{c}_5 - 41\bar{c}_6 + 15\bar{c}_7 &\geq 0 \\ +12\bar{c}_3 + 12\bar{c}_4 + 12\bar{c}_5 + 12\bar{c}_6 - 36\bar{c}_7 &\geq 0 \end{aligned}$$

The vector  $[7, 5, 12, 12, 10, 14, 16]$  is feasible for  $\widetilde{P_c}(\langle \mathcal{S} \rangle)$ .  $\square$

**Proposition 10.**  *$\langle \mathcal{S} \rangle$  is a Non-FC-family.*

*Proof.* We exhibit a system of linear inequalities in  $\mathbb{R}_{\geq 0}^7$  that is infeasible. As a certificate of infeasibility we display Farkas dual values in square brackets before each inequality. Structurally, we see that the only difference between the UC families that generated this system of linear inequalities and the previous one are the red inequalities. In contrast to the other inequalities, the red one here is derived from the following UC family:  $\{S_{[n] \setminus \{3\} \setminus \{4\}} \uplus \langle \mathcal{S} \rangle\} \cup \{\{1, 3, 4\}, \{1, 3, 4, 5\}\}$ .



$$\begin{aligned}
[1] : c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 &\geq 1 \\
[19] : -52c_1 + 4c_2 + 12c_3 + 12c_4 + 4c_6 &\geq 0 \\
[2] : +6c_1 - 54c_2 + 10c_3 + 10c_4 + 2c_6 + 2c_7 &\geq 0 \\
[109] : +8c_1 - 8c_3 - 8c_4 + 8c_7 &\geq 0 \\
[16] : -48c_5 + 16c_6 + 16c_7 &\geq 0 \\
[20] : +5c_1 + 1c_2 + 7c_3 + 7c_4 + 13c_5 - 41c_6 + 15c_7 &\geq 0 \\
[40] : +12c_3 + 12c_4 + 12c_5 + 12c_6 - 36c_7 &\geq 0
\end{aligned}$$

□

**Corollary 5.** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements such that  $\emptyset \in \mathcal{A}$ . The feasibility of  $\widetilde{P}_c(\mathcal{A})$  does not imply the feasibility of  $P_c$ .*

**Corollary 6.** *Let  $\mathcal{A}$  be an UC family that covers  $n$  elements such that  $\emptyset \in \mathcal{A}$ .  $NUM(\mathcal{A}) \leq 1$  does not imply that  $P_c$  is feasible.*

## Conclusion

In this work we design a cutting-plane algorithm that determines if a given UC family necessarily implies Frankl's conjecture for all families that contain it. By employing exact rational integer programming and highly redundant verification routines, we classify more previously unknown minimal non-isomorphic FC-families than the total output of the past twenty-five years of research on the topic. The effects of *safely* automating the discovery of FC-families allow us to answer several open questions of Morris [13] and Vaughan [18]. In particular, the counterexamples we exhibit to settle open questions of interest require no *trust* from the reader, in the sense that they are independent of the complex optimization processes that led to them, and can be checked by hand. Furthermore, our framework can be used to improve several other results in the following ways:

- Since our algorithm determines exactly whether a given UC family  $\mathcal{A}$  that covers  $n$  elements is FC or Non-FC for  $6 \leq n \leq 10$ , lower bounds for previously unknown  $FC(k, n)$  in this range become trivial to obtain. Coupled with a computer algebra system or graph isomorphism software to obtain the isomorphism types of generators, upper or exact bounds for previously unknown  $FC(k, n)$  can be easily obtained in the aforementioned range.
- The approach of Morris [13] for the classification of FC-families on five elements lends itself well to being generalized within our framework. The number of minimal non-isomorphic generators for FC-families seems to quickly grow for  $n \geq 6$ , but we believe a complete classification for  $n = 6$  is possible with routine work.
- Proving the 3-sets conjecture of Morris [13], by recovering the arguments of Vaughan [19] through a classification of  $FC(3, n)$  for  $7 \leq n \leq 9$  and using Morris's lower bound on 3-sets, is within reach.

- Each FC-family represents an “optimality” cut for the integer program that relates to Frankl’s conjecture in the work of Pulaj, Raymond and Theis [15]. Improved bounds may be achieved, if an efficient separation routine is implemented for UC inequalities on  $n = 13$ . Alternatively, a counterexample may be found.

Finally, we believe that by separating UC and FS inequalities, our framework can possibly classify FC-families for  $11 \leq n \leq 14$ . For larger  $n$  other advanced techniques such as column generation may be necessary. More generally, we believe our present work underscores the importance of bringing together computational tools and problems from different fields that rarely cross paths – despite being well-suited for each other – as is often the case with advanced integer programming techniques and fitting problems in extremal combinatorics. This is particularly useful when the problem at hand lacks apparent structure and even finding interesting examples becomes non-trivial.

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## 6 Appendix

To check the claims of *infeasibility* for the linear systems in this paper it is sufficient to ensure that the vector of values exhibited in square brackets before each row corresponds to the vector  $y$  below.

**Theorem 6 (Farkas' Lemma).** *Let  $A_1 \in \mathbb{R}^{m_1 \times n}$ ,  $A_2 \in \mathbb{R}^{m_2 \times n}$  and  $A_3 \in \mathbb{R}^{m_3 \times n}$ . Also let  $b_1 \in \mathbb{R}^{m_1}$ ,  $b_2 \in \mathbb{R}^{m_2}$  and  $b_3 \in \mathbb{R}^{m_3}$ . Then the following system of linear equalities and inequalities in  $x \in \mathbb{R}^n$  :*

$$\begin{aligned}
 A_1 x &= b_1 \\
 A_2 x &\leq b_2 \\
 A_3 x &\geq b_3 \\
 x &\geq 0
 \end{aligned}$$

*is infeasible if and only if there exist  $y_1 \in \mathbb{R}^{m_1}$ ,  $y_2 \in \mathbb{R}^{m_2}$ ,  $y_3 \in \mathbb{R}^{m_3}$  such that:*

$$\begin{aligned}
 b_1^\top y_1 + b_2^\top y_2 + b_3^\top y_3 &> 0 \\
 A_1^\top y_1 + A_2^\top y_2 + A_3^\top y_3 &\leq 0 \\
 y_2 &\leq 0 \\
 y_3 &\geq 0
 \end{aligned}$$

*Proof of proposition 6.* We identify sets in  $S_6$  with the columns in the matrix below. For each column, the number on the top row represents its corresponding variable index in  $FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n$ . Column  $W$  corresponds to a weight vector for the elements in  $[n]$ . The columns representing families of sets  $\mathcal{S}'$  and  $\mathcal{T}$  are colored red and blue, respectively. As previously,  $\langle \mathcal{S}' \rangle = \mathcal{S}' \cup \mathcal{T}$ .

$W$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
20	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
17	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
49	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
57	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
59	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
60	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
61	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
62	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
63	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

We prove that  $FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n$  is infeasible by branching on  $x_0$  and showing that the linear relaxations of the two subproblems are infeasible. We do this by explicitly exhibiting Farkas dual values (shown in square brackets) for each row of some irreducible infeasible subset of constraints. Define  $FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n^1 := FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n \cap (x_0 = 1)$ ,  $FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n^0 := FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n \cap (x_0 = 0)$ . Then it is clear that if  $FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n^1$  and  $FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n^0$  are infeasible, this implies  $FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n$  is infeasible. Let  $FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n^1 \supseteq FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n^1$ , and let the following be the linear relaxation of  $FC(\mathcal{A}, \langle \mathcal{S}' \rangle)_n^1$ , defined by the following constraints (trivial ones not shown):

1.  $[44] : x_0 = 1$ .
2. UC inequalities:
  - $[-2] : x_{11} + x_{45} - x_9 \leq 1$ ,  $[-3] : x_{13} + x_{59} - x_9 \leq 1$ ,
  - $[-2] : x_{14} + x_{43} - x_{10} \leq 1$ ,  $[-1] : x_{22} + x_{61} - x_{20} \leq 1$ ,
  - $[-3] : x_{23} + x_{60} - x_{20} \leq 1$ ,  $[-1] : x_{35} + x_{45} - x_{33} \leq 1$ ,
  - $[-3] : x_{35} + x_{62} - x_{34} \leq 1$ ,  $[-6] : x_{37} + x_{59} - x_{33} \leq 1$ ,
  - $[-1] : x_{38} + x_{43} - x_{34} \leq 1$ ,  $[-3] : x_{38} + x_{45} - x_{36} \leq 1$ ,
  - $[-1] : x_{38} + x_{61} - x_{36} \leq 1$ ,  $[-1] : x_{39} + x_{44} - x_{36} \leq 1$ ,
  - $[-2] : x_{42} + x_{55} - x_{34} \leq 1$ ,  $[-2] : x_{53} + x_{43} - x_{33} \leq 1$ ,
  - $[-5] : x_{54} + x_{43} - x_{34} \leq 1$ ,  $[-3] : x_{44} + x_{55} - x_{36} \leq 1$ ,
  - $[-4] : x_{47} + x_{49} - x_{33} \leq 1$ .
3. FS inequalities:
  - $[0] : x_{47} - x_1 \leq 0$ ,  $[-6] : x_{63} - x_1 \leq 0$ ,  $[-14] : x_{63} - x_8 \leq 0$ ,
  - $[-1] : x_7 - x_4 \leq 0$ ,  $[-16] : x_{55} - x_4 \leq 0$ ,  $[-12] : x_{63} - x_{12} \leq 0$ ,
  - $[-3] : x_{14} - x_2 \leq 0$ ,  $[-24] : x_{46} - x_2 \leq 0$ ,  $[-12] : x_{47} - x_3 \leq 0$ ,
  - $[-21] : x_{61} - x_{17} \leq 0$ ,  $[-19] : x_{62} - x_{18} \leq 0$ ,  $[-4] : x_{63} - x_{19} \leq 0$ ,
  - $[-24] : x_{31} - x_{24} \leq 0$ ,  $[-1] : x_{37} - x_{32} \leq 0$ ,  $[-4] : x_{38} - x_{32} \leq 0$ ,
  - $[-23] : x_{39} - x_{32} \leq 0$ ,  $[-16] : x_{47} - x_{40} \leq 0$ ,  $[-11] : x_{55} - x_{48} \leq 0$ ,
  - $[-8] : x_{63} - x_{56} \leq 0$ .

4. FC inequalities:

$$\begin{aligned}
[-2] : x_{15} + x_{53} - x_1 - x_5 &\leq 0, [-7] : x_{15} + x_{57} - x_1 - x_9 \leq 0, \\
[-9] : x_{58} + x_{15} - x_8 - x_{10} &\leq 0, [-2] : x_{15} + x_{59} - x_1 - x_{11} \leq 0, \\
[-7] : x_{45} + x_{23} - x_1 - x_5 &\leq 0, [-5] : x_{60} + x_{23} - x_{20} - x_{16} \leq 0, \\
[-1] : x_{61} + x_{23} - x_{21} - x_{16} &\leq 0, [-5] : x_{45} + x_{27} - x_1 - x_9 \leq 0, \\
[-3] : x_{27} + x_{61} - x_{25} - x_{16} &\leq 0, [0] : x_{43} + x_{29} - x_1 - x_9 \leq 0, \\
[-5] : x_{54} + x_{29} - x_{20} - x_{16} &\leq 0, [-6] : x_{29} + x_{59} - x_{16} - x_{25} \leq 0, \\
[-4] : x_{43} + x_{30} - x_{10} - x_8 &\leq 0, [-2] : x_{53} + x_{30} - x_{16} - x_{20} \leq 0, \\
[-7] : x_{59} + x_{30} - x_{16} - x_{26} &\leq 0, [-4] : x_{31} + x_{60} - x_{16} - x_{28} \leq 0, \\
[-1] : x_{43} + x_{45} - x_8 - x_{41} &\leq 0, [-1] : x_{43} + x_{62} - x_8 - x_{42} \leq 0, \\
[-3] : x_{47} + x_{62} - x_8 - x_{46} &\leq 0, [-3] : x_{51} + x_{62} - x_{16} - x_{50} \leq 0, \\
[-7] : x_7 + x_{54} - x_4 - x_6 &\leq 0, [-9] : x_{51} + x_{53} - x_{48} - x_{49} \leq 0.
\end{aligned}$$

5. Feasibility inequality:

$$\begin{aligned}
[-0.5] : 88x_0 + 58x_1 + 54x_2 + 24x_3 + 48x_4 + 18x_5 + 14x_6 - 16x_7 + 64x_8 \\
+ 34x_9 + 30x_{10} + 24x_{12} - 6x_{13} - 10x_{14} - 40x_{15} + 72x_{16} + 42x_{17} \\
+ 38x_{18} + 8x_{19} + 32x_{20} + 2x_{21} - 2x_{22} - 32x_{23} + 48x_{24} + 18x_{25} \\
+ 14x_{26} - 16x_{27} + 8x_{28} - 22x_{29} - 26x_{30} - 56x_{31} + 56x_{32} + 26x_{33} \\
+ 22x_{34} - 8x_{35} + 16x_{36} - 14x_{37} - 18x_{38} - 48x_{39} + 32x_{40} + 2x_{41} \\
- 2x_{42} - 32x_{43} - 8x_{44} - 38x_{45} - 42x_{46} - 72x_{47} + 40x_{48} \\
+ 10x_{49} + 6x_{50} - 24x_{51} - 30x_{53} - 34x_{54} - 64x_{55} + 16x_{56} - 14x_{57} \\
- 18x_{58} - 48x_{59} - 24x_{60} - 54x_{61} - 58x_{62} - 88x_{63} \leq -1.
\end{aligned}$$

Let  $\overline{FC(\mathcal{A}, \mathcal{S}')_n^0} \supseteq FC(\mathcal{A}, \mathcal{S}')_n^0$ , and let the following be the linear relaxation of  $\overline{FC(\mathcal{A}, \mathcal{S}')_n^0}$ , defined by the following constraints (trivial ones not shown):

1.  $[-186.5] : x_0 = 0$

2. FS inequalities:

$$\begin{aligned}
[-7.5] : x_1 - x_0 &\leq 0, [-10] : x_6 - x_0 \leq 0, [-8.5] : x_{11} - x_0 \leq 0, \\
[0] : x_{19} - x_0 &\leq 0, [-8.5] : x_{23} - x_0 \leq 0, [-4] : x_{35} - x_0 \leq 0, \\
[-7] : x_{37} - x_0 &\leq 0, [-9] : x_{38} - x_0 \leq 0, [-24] : x_{39} - x_0 \leq 0, \\
[-2.5] : x_{41} - x_0 &\leq 0, [-16] : x_{44} - x_0 \leq 0, [-21] : x_{46} - x_0 \leq 0, \\
[-15] : x_{47} - x_0 &\leq 0, [-6] : x_{50} - x_0 \leq 0, [-19] : x_{55} - x_0 \leq 0, \\
[-5.5] : x_{56} - x_0 &\leq 0, [-17] : x_{59} - x_0 \leq 0, [-16] : x_{61} - x_0 \leq 0, \\
[-11] : x_{62} - x_0 &\leq 0, [-23] : x_{63} - x_0 \leq 0, [-12] : x_{13} - x_{12} \leq 0, \\
[-12.5] : x_{14} - x_2 &\leq 0, [-12.5] : x_{22} - x_{18} \leq 0, [-6.5] : x_{62} - x_{18} \leq 0, \\
[-1] : x_{42} - x_{40} &\leq 0, [-7] : x_{51} - x_{48} \leq 0, [-7.5] : x_{43} - x_8 \leq 0, \\
[-5.5] : x_{29} - x_{17} &\leq 0, [-5.5] : x_{61} - x_9 \leq 0, [0] : x_{63} - x_{56} \leq 0.
\end{aligned}$$

3. FC inequalities:

$$\begin{aligned}
[-7.5] : x_{15} + x_{45} - x_1 - x_{13} &\leq 0, [-9] : x_{15} + x_{53} - x_1 - x_5 \leq 0, \\
[-3.5] : x_{15} + x_{57} - x_1 - x_9 &\leq 0, [-7.5] : x_{23} + x_{62} - x_{16} - x_{22} \leq 0, \\
[-8] : x_{27} + x_{45} - x_1 - x_9 &\leq 0, [-8.5] : x_{31} + x_{43} - x_1 - x_{11} \leq 0, \\
[-1] : x_{31} + x_{53} - x_{16} - x_{21} &\leq 0, [-3.5] : x_{45} + x_{57} - x_8 - x_{41} \leq 0, \\
[-9] : x_{55} + x_{58} - x_{16} - x_{50} &\leq 0, [-17] : x_7 + x_{54} - x_4 - x_6 \leq 0, \\
[-5] : x_{51} + x_{53} - x_{48} - x_{49} &\leq 0.
\end{aligned}$$

4. FC-chain inequalities (it is easy to check that the explicit chains work for all subsets):

$$\begin{aligned}
& [-1.5] : \mathbf{x}_{29} + \mathbf{x}_{47} + \mathbf{x}_{61} + \mathbf{x}_{63} - \mathbf{x}_8 - \mathbf{x}_{13} - \mathbf{x}_{17} - \mathbf{x}_{56} \leq 0, \\
& (29 \xrightarrow{19^{FS}} 17), (63 \xrightarrow{8^{FS}} 8), (47 \xrightarrow{8^{FS}} 8), (61 \xrightarrow{56^{FS}} 56), (63 \xrightarrow{56^{FS}} 56), \\
& (47 \xrightarrow{29^{UC}} 13), (61 \xrightarrow{19^{FS}} 17). \\
& [-4] : \mathbf{x}_{29} + \mathbf{x}_{61} + \mathbf{x}_{62} - \mathbf{x}_{16} - \mathbf{x}_{17} - \mathbf{x}_{28} \leq 0, \\
& (29 \xrightarrow{16^{FS}} 16), (61 \xrightarrow{19^{FS}} 17), (62 \xrightarrow{19^{FS}} 18 \xrightarrow{16^{FS}} 16), \\
& (29 \xrightarrow{62^{UC}} 28). \\
& [-7.5] : \mathbf{x}_{30} + \mathbf{x}_{31} + \mathbf{x}_{47} + \mathbf{x}_{63} - \mathbf{x}_8 - \mathbf{x}_{14} - \mathbf{x}_{16} - \mathbf{x}_{24} \leq 0, \\
& (63 \xrightarrow{8^{FS}} 8), (47 \xrightarrow{8^{FS}} 8), (63 \xrightarrow{16^{FS}} 16), (47 \xrightarrow{30^{UC}} 14), (30 \xrightarrow{16^{FS}} 16), \\
& (30 \xrightarrow{56^{FS}} 24), (31 \xrightarrow{16^{FS}} 16), (31 \xrightarrow{56^{FS}} 24). \\
& [-4] : \mathbf{x}_{30} + \mathbf{x}_{31} + \mathbf{x}_{55} - \mathbf{x}_{19} - \mathbf{x}_{22} - \mathbf{x}_{24} \leq 0, \\
& (30 \xrightarrow{56^{FS}} 24), (31 \xrightarrow{56^{FS}} 24), (31 \xrightarrow{19^{FS}} 19), (55 \xrightarrow{30^{UC}} 22), (55 \xrightarrow{19^{FS}} 19). \\
& [-7] : \mathbf{x}_{30} + \mathbf{x}_{31} + \mathbf{x}_{59} - \mathbf{x}_{16} - \mathbf{x}_{24} - \mathbf{x}_{26} \leq 0, \\
& (30 \xrightarrow{56^{FS}} 24), (31 \xrightarrow{56^{FS}} 24), (31 \xrightarrow{16^{FS}} 16), (30 \xrightarrow{16^{FS}} 16), \\
& (59 \xrightarrow{16^{FS}} 16), (59 \xrightarrow{30^{UC}} 26). \\
& [0] : \mathbf{x}_{47} + \mathbf{x}_{54} + \mathbf{x}_{63} - \mathbf{x}_8 - \mathbf{x}_{16} - \mathbf{x}_{38} \leq 0, \\
& (63 \xrightarrow{8^{FS}} 8), (63 \xrightarrow{16^{FS}} 16), (47 \xrightarrow{8^{FS}} 8), (54 \xrightarrow{16^{FS}} 16), (54 \xrightarrow{47^{UC}} 38). \\
& [-12] : \mathbf{x}_{47} + \mathbf{x}_{60} + \mathbf{x}_{63} - \mathbf{x}_8 - \mathbf{x}_{44} - \mathbf{x}_{56} \leq 0. \\
& (47 \xrightarrow{8^{FS}} 8), (63 \xrightarrow{8^{FS}} 8), (63 \xrightarrow{56^{FS}} 56), (60 \xrightarrow{56^{FS}} 56), (60 \xrightarrow{47^{UC}} 44).
\end{aligned}$$

5. Feasibility inequality:

$$\begin{aligned}
& [-0.5] : 58\mathbf{x}_1 + 54\mathbf{x}_2 + 24\mathbf{x}_3 + 48\mathbf{x}_4 + 18\mathbf{x}_5 + 14\mathbf{x}_6 - 16\mathbf{x}_7 + 64\mathbf{x}_8 \\
& + 34\mathbf{x}_9 + 30\mathbf{x}_{10} + 24\mathbf{x}_{12} - 6\mathbf{x}_{13} - 10\mathbf{x}_{14} - 40\mathbf{x}_{15} + 72\mathbf{x}_{16} + 42\mathbf{x}_{17} \\
& + 38\mathbf{x}_{18} + 8\mathbf{x}_{19} + 32\mathbf{x}_{20} + 2\mathbf{x}_{21} - 2\mathbf{x}_{22} - 32\mathbf{x}_{23} + 48\mathbf{x}_{24} + 18\mathbf{x}_{25} \\
& + 14\mathbf{x}_{26} - 16\mathbf{x}_{27} + 8\mathbf{x}_{28} - 22\mathbf{x}_{29} - 26\mathbf{x}_{30} - 56\mathbf{x}_{31} + 56\mathbf{x}_{32} + 26\mathbf{x}_{33} \\
& + 22\mathbf{x}_{34} - 8\mathbf{x}_{35} + 16\mathbf{x}_{36} - 14\mathbf{x}_{37} - 18\mathbf{x}_{38} - 48\mathbf{x}_{39} + 32\mathbf{x}_{40} + 2\mathbf{x}_{41} \\
& - 2\mathbf{x}_{42} - 32\mathbf{x}_{43} - 8\mathbf{x}_{44} - 38\mathbf{x}_{45} - 42\mathbf{x}_{46} - 72\mathbf{x}_{47} + 40\mathbf{x}_{48} \\
& + 10\mathbf{x}_{49} + 6\mathbf{x}_{50} - 24\mathbf{x}_{51} - 30\mathbf{x}_{53} - 34\mathbf{x}_{54} - 64\mathbf{x}_{55} + 16\mathbf{x}_{56} - 14\mathbf{x}_{57} \\
& - 18\mathbf{x}_{58} - 48\mathbf{x}_{59} - 24\mathbf{x}_{60} - 54\mathbf{x}_{61} - 58\mathbf{x}_{62} - 88\mathbf{x}_{63} \leq -1.
\end{aligned}$$

□

In order to improve readability, in the following tables we do not use curly brackets when denoting families of sets and the sets themselves. For example for  $\{\{1, 2, 3\}, \{2, 3, 4\}\}$ , we simply write  $[123, 234]$ .

Previously unknown minimal* non-isomorphic generators for FC-families on [6]	
[1256, 3456, 456, 236]*	$1 \mapsto 7, 2 \mapsto 15, 3 \mapsto 15, 4 \mapsto 11, 5 \mapsto 14, 6 \mapsto 20$
[12456, 2346, 456, 356]*	$1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 5, 4 \mapsto 5, 5 \mapsto 6, 6 \mapsto 7$
[12345, 1356, 456, 356]*	$1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 4, 4 \mapsto 4, 5 \mapsto 5, 6 \mapsto 5$
[12345, 2346, 456, 236]*	$1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 3, 5 \mapsto 4, 6 \mapsto 5$
[12345, 2346, 456, 236]*	$1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 3, 5 \mapsto 4, 6 \mapsto 5$
[12346, 1256, 456, 356]*	$1 \mapsto 4, 2 \mapsto 4, 3 \mapsto 7, 4 \mapsto 7, 5 \mapsto 9, 6 \mapsto 10$
[12356, 1345, 456, 236]*	$1 \mapsto 8, 2 \mapsto 12, 3 \mapsto 16, 4 \mapsto 15, 5 \mapsto 17, 6 \mapsto 20$
[12356, 1234, 456, 356]*	$1 \mapsto 8, 2 \mapsto 8, 3 \mapsto 24, 4 \mapsto 24, 5 \mapsto 27, 6 \mapsto 29$
[12456, 1356, 456, 326]*	$1 \mapsto 45, 2 \mapsto 71, 3 \mapsto 77, 4 \mapsto 59, 5 \mapsto 74, 6 \mapsto 103$
[136, 2456, 3456, 456, 123]*	$1 \mapsto 6, 2 \mapsto 5, 3 \mapsto 7, 4 \mapsto 3, 5 \mapsto 3, 6 \mapsto 6$
[136, 1256, 3456, 456, 123]*	$1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 1, 5 \mapsto 1, 6 \mapsto 2$
[2346, 3456, 2456, 2356, 1234]*	$1 \mapsto 2, 2 \mapsto 5, 3 \mapsto 5, 4 \mapsto 5, 5 \mapsto 4, 6 \mapsto 5$
[3456, 2456, 2356, 1346, 1246, 1234]*	$1 \mapsto 3, 2 \mapsto 4, 3 \mapsto 4, 4 \mapsto 4, 5 \mapsto 3, 6 \mapsto 4$
[3456, 2456, 2356, 1346, 1245, 1234]*	$1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 2, 4 \mapsto 2, 5 \mapsto 2, 6 \mapsto 2$
[3456, 2456, 1456, 1236, 1235, 1234]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1, 4 \mapsto 1, 5 \mapsto 1, 6 \mapsto 1$
[3456, 2456, 1356, 1246, 1235, 1234]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1, 4 \mapsto 1, 5 \mapsto 1, 6 \mapsto 1$
[3456, 2456, 2356, 2346, 1456, 1356]*	$1 \mapsto 8, 2 \mapsto 14, 3 \mapsto 15, 4 \mapsto 15, 5 \mapsto 16, 6 \mapsto 19$
[3456, 2456, 2356, 2346, 1456, 1236]*	$1 \mapsto 3, 2 \mapsto 4, 3 \mapsto 4, 4 \mapsto 4, 5 \mapsto 4, 6 \mapsto 5$
[3456, 2456, 2356, 1456, 1356, 1234]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1, 4 \mapsto 1, 5 \mapsto 1, 6 \mapsto 1$
[3456, 2456, 2356, 1456, 1346, 1245]*	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 2, 4 \mapsto 3, 5 \mapsto 3, 6 \mapsto 3$
[3456, 2456, 2356, 1456, 1346, 1235]*	$1 \mapsto 5, 2 \mapsto 4, 3 \mapsto 5, 4 \mapsto 5, 5 \mapsto 6, 6 \mapsto 6$
[3456, 2456, 2356, 1456, 1236, 1235]*	$1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 3, 4 \mapsto 2, 5 \mapsto 3, 6 \mapsto 3$
[3456, 2456, 2356, 1456, 1236, 1234]*	$1 \mapsto 4, 2 \mapsto 5, 3 \mapsto 5, 4 \mapsto 5, 5 \mapsto 4, 6 \mapsto 5$
[3456, 2456, 2356, 1346, 1345, 1246]*	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 3, 5 \mapsto 3, 6 \mapsto 3$
[3456, 2456, 2356, 1346, 1246, 1235]*	$1 \mapsto 3, 2 \mapsto 4, 3 \mapsto 4, 4 \mapsto 3, 5 \mapsto 4, 6 \mapsto 4$
[12346, 3456, 2456, 2356, 1456, 1356, 1256]*	$1 \mapsto 5, 2 \mapsto 5, 3 \mapsto 5, 4 \mapsto 5, 5 \mapsto 6, 6 \mapsto 7$
[1236, 3456, 2456, 2356, 1456, 1356, 1246]*	$2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 3, 4 \mapsto 3, 5 \mapsto 3, 6 \mapsto 4$
[1456, 3456, 2456, 2356, 1346, 1246, 1236]*	$2 \mapsto 3, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 3, 5 \mapsto 3, 6 \mapsto 4$
[1256, 3456, 2456, 2356, 1456, 1346, 1236]*	$2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 3, 4 \mapsto 3, 5 \mapsto 3, 6 \mapsto 4$
[2356, 2456, 345, 13456, 12346]*	$2 \mapsto 3, 2 \mapsto 8, 3 \mapsto 12, 4 \mapsto 12, 5 \mapsto 13, 6 \mapsto 9$
[1234, 1256, 246, 23456, 13456]*	$2 \mapsto 9, 2 \mapsto 12, 3 \mapsto 7, 4 \mapsto 11, 5 \mapsto 7, 6 \mapsto 11$
[1236, 2456, 125, 23456, 13456]*	$2 \mapsto 32, 2 \mapsto 34, 3 \mapsto 19, 4 \mapsto 16, 5 \mapsto 32, 6 \mapsto 25$
[1246, 1256, 123, 23456, 13456]*	$2 \mapsto 7, 2 \mapsto 7, 3 \mapsto 6, 4 \mapsto 3, 5 \mapsto 4, 6 \mapsto 5$

Previously unknown minimal* non-isomorphic generators for FC-families on [7]	
[3457, 567, 467, 123]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 5, 5 \mapsto 5, 6 \mapsto 6, 7 \mapsto 7$
[2467, 567, 347, 126]*	$1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 1, 4 \mapsto 2, 5 \mapsto 2, 6 \mapsto 3, 7 \mapsto 3$
[357, 367, 4567, 1237]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 6, 4 \mapsto 2, 5 \mapsto 5, 6 \mapsto 5, 7 \mapsto 7$
[356, 367, 4567, 1237]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 4, 4 \mapsto 1, 5 \mapsto 3, 6 \mapsto 4, 7 \mapsto 3$
[257, 367, 4567, 1237]*	$1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 2, 4 \mapsto 1, 5 \mapsto 2, 6 \mapsto 2, 7 \mapsto 3$
[256, 367, 4567, 1237]*	$1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 2, 4 \mapsto 1, 5 \mapsto 3, 6 \mapsto 4, 7 \mapsto 3$
[346, 367, 4567, 1237]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 4, 4 \mapsto 3, 5 \mapsto 3, 6 \mapsto 4, 7 \mapsto 2$
[245, 367, 4567, 1237]*	$1 \mapsto 2, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 2, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 4$
[246, 367, 4567, 1237]*	$1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 3, 4 \mapsto 3, 5 \mapsto 3, 6 \mapsto 4, 7 \mapsto 1$
[235, 367, 4567, 1237]*	$1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 1, 5 \mapsto 3, 6 \mapsto 4, 7 \mapsto 1$
[234, 367, 4567, 1237]*	$1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 7, 4 \mapsto 5, 5 \mapsto 5, 6 \mapsto 5, 7 \mapsto 4$
[12456, 34567, 267, 127]*	$1 \mapsto 78, 2 \mapsto 105, 3 \mapsto 16, 4 \mapsto 27, 5 \mapsto 27, 6 \mapsto 84, 7 \mapsto 103$
[12456, 34567, 267, 257]*	$1 \mapsto 1, 2 \mapsto 9, 3 \mapsto 1, 4 \mapsto 2, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 9$
[3467, 4567, 2367, 2345, 1357]*	$1 \mapsto 20, 2 \mapsto 36, 3 \mapsto 52, 4 \mapsto 45, 5 \mapsto 46, 6 \mapsto 39, 7 \mapsto 49$
[3456, 4567, 2367, 1357, 1247]*	$1 \mapsto 15, 2 \mapsto 15, 3 \mapsto 20, 4 \mapsto 18, 5 \mapsto 19, 6 \mapsto 19, 7 \mapsto 23$
[3456, 4567, 2367, 1357, 1246]*	$1 \mapsto 17, 2 \mapsto 16, 3 \mapsto 22, 4 \mapsto 19, 5 \mapsto 21, 6 \mapsto 24, 7 \mapsto 22$
[2347, 4567, 3567, 1267, 1245]*	$1 \mapsto 69, 2 \mapsto 91, 3 \mapsto 71, 4 \mapsto 93, 5 \mapsto 87, 6 \mapsto 81, 7 \mapsto 103$
[2346, 4567, 3567, 2347, 1267]*	$1 \mapsto 6, 2 \mapsto 13, 3 \mapsto 14, 4 \mapsto 14, 5 \mapsto 9, 6 \mapsto 16, 7 \mapsto 16$
[2345, 4567, 3567, 1247, 1236]*	$1 \mapsto 24, 2 \mapsto 32, 3 \mapsto 33, 4 \mapsto 33, 5 \mapsto 32, 6 \mapsto 31, 7 \mapsto 31$
[2345, 4567, 2367, 1357, 1247]*	$1 \mapsto 3, 2 \mapsto 4, 3 \mapsto 4, 4 \mapsto 4, 5 \mapsto 4, 6 \mapsto 3, 7 \mapsto 4$
[2345, 4567, 2367, 1357, 1246]*	$1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 2, 4 \mapsto 2, 5 \mapsto 2, 6 \mapsto 2, 7 \mapsto 2$
[1356, 4567, 2367, 2345, 1357]*	$1 \mapsto 5, 2 \mapsto 6, 3 \mapsto 9, 4 \mapsto 6, 5 \mapsto 9, 6 \mapsto 8, 7 \mapsto 8$
[12456, 13457, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 11, 2 \mapsto 12, 3 \mapsto 11, 4 \mapsto 13, 5 \mapsto 13, 6 \mapsto 14, 7 \mapsto 15$
[12345, 13457, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 12, 2 \mapsto 12, 3 \mapsto 13, 4 \mapsto 13, 5 \mapsto 13, 6 \mapsto 12, 7 \mapsto 15$
[12345, 23456, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 6, 2 \mapsto 7, 3 \mapsto 7, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 8, 7 \mapsto 8$
[12345, 13456, 23457, 12367, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 6, 5 \mapsto 6, 6 \mapsto 7, 7 \mapsto 7$
[23456, 12457, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 6, 2 \mapsto 7, 3 \mapsto 7, 4 \mapsto 8, 5 \mapsto 8, 6 \mapsto 8, 7 \mapsto 8$
[12356, 12457, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 8, 2 \mapsto 8, 3 \mapsto 8, 4 \mapsto 8, 5 \mapsto 9, 6 \mapsto 9, 7 \mapsto 10$
[23456, 14567, 13567, 13467, 13457, 13456, 12567, 12467, 12457, 12456, 12367, 12357, 12356, 12347]*	$1 \mapsto 14, 2 \mapsto 12, 3 \mapsto 12, 4 \mapsto 11, 5 \mapsto 12, 6 \mapsto 12, 7 \mapsto 11$
[23456, 14567, 13567, 13467, 13457, 13456, 12567, 12467, 12457, 12456, 12367, 12357, 12346, 12345]*	$1 \mapsto 7, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 6, 5 \mapsto 6, 6 \mapsto 6, 7 \mapsto 5$



Previously unknown minimal* non-isomorphic generators for FC-families on [7]	
[23456, 12357, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 7, 2 \mapsto 10, 3 \mapsto 10, 4 \mapsto 10, 5 \mapsto 11, 6 \mapsto 11, 7 \mapsto 12$
[12456, 12357, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 9, 2 \mapsto 9, 3 \mapsto 8, 4 \mapsto 9, 5 \mapsto 10, 6 \mapsto 10, 7 \mapsto 11$
[12346, 12357, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 12, 2 \mapsto 12, 3 \mapsto 13, 4 \mapsto 13, 5 \mapsto 12, 6 \mapsto 13, 7 \mapsto 15$
[13456, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 7$
[12456, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 7$
[12356, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 7, 2 \mapsto 7, 3 \mapsto 8, 4 \mapsto 8, 5 \mapsto 8, 6 \mapsto 9, 7 \mapsto 9$
[12345, 23456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 7, 2 \mapsto 8, 3 \mapsto 8, 4 \mapsto 9, 5 \mapsto 9, 6 \mapsto 9, 7 \mapsto 9$
[12356, 12456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 6, 5 \mapsto 6, 6 \mapsto 7, 7 \mapsto 7$
[12345, 12456, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 7$
[12346, 12356, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 6, 5 \mapsto 6, 6 \mapsto 7, 7 \mapsto 7$
[12345, 12356, 13457, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 9, 2 \mapsto 9, 3 \mapsto 9, 4 \mapsto 9, 5 \mapsto 10, 6 \mapsto 10, 7 \mapsto 10$
[23456, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 5, 2 \mapsto 7, 3 \mapsto 7, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 8$
[13456, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 7, 4 \mapsto 7, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 8$
[12356, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 6, 2 \mapsto 7, 3 \mapsto 7, 4 \mapsto 6, 5 \mapsto 7, 6 \mapsto 7, 7 \mapsto 8$
[12345, 12347, 12357, 23457, 12467, 13467, 23467, 12567, 13567, 23567, 14567, 24567, 34567]*	$1 \mapsto 11, 2 \mapsto 12, 3 \mapsto 12, 4 \mapsto 12, 5 \mapsto 12, 6 \mapsto 11, 7 \mapsto 14$

Previously unknown minimal* non-isomorphic generators for FC-families on [7]	
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12367, 12347]*	$1 \mapsto 7, 2 \mapsto 8, 3 \mapsto 9, 4 \mapsto 9, 5 \mapsto 8, 6 \mapsto 9, 7 \mapsto 9$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12367, 12346]*	$1 \mapsto 7, 2 \mapsto 8, 3 \mapsto 9, 4 \mapsto 9, 5 \mapsto 8, 6 \mapsto 9, 7 \mapsto 9$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12367, 12345]*	$1 \mapsto 3, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 4, 5 \mapsto 4, 6 \mapsto 4, 7 \mapsto 4$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12357, 12347]*	$1 \mapsto 7, 2 \mapsto 8, 3 \mapsto 9, 4 \mapsto 9, 5 \mapsto 9, 6 \mapsto 8, 7 \mapsto 9$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12357, 12346]*	$1 \mapsto 3, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 4, 5 \mapsto 4, 6 \mapsto 4, 7 \mapsto 4$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12357, 12345]*	$1 \mapsto 7, 2 \mapsto 8, 3 \mapsto 9, 4 \mapsto 9, 5 \mapsto 9, 6 \mapsto 8, 7 \mapsto 9$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12347, 12346]*	$1 \mapsto 7, 2 \mapsto 8, 3 \mapsto 9, 4 \mapsto 9, 5 \mapsto 8, 6 \mapsto 9, 7 \mapsto 9$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12347, 12345]*	$1 \mapsto 7, 2 \mapsto 8, 3 \mapsto 9, 4 \mapsto 9, 5 \mapsto 9, 6 \mapsto 8, 7 \mapsto 9$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12457, 12346, 12345]*	$1 \mapsto 7, 2 \mapsto 8, 3 \mapsto 9, 4 \mapsto 9, 5 \mapsto 9, 6 \mapsto 9, 7 \mapsto 8$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 13456, 12347, 12346, 12345]*	$1 \mapsto 7, 2 \mapsto 7, 3 \mapsto 9, 4 \mapsto 9, 5 \mapsto 8, 6 \mapsto 8, 7 \mapsto 8$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 12457, 12456, 12347, 12346]*	$1 \mapsto 8, 2 \mapsto 9, 3 \mapsto 9, 4 \mapsto 10, 5 \mapsto 9, 6 \mapsto 9, 7 \mapsto 9$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13467, 12457, 12456, 12347, 12345]*	$1 \mapsto 8, 2 \mapsto 9, 3 \mapsto 9, 4 \mapsto 10, 5 \mapsto 9, 6 \mapsto 9, 7 \mapsto 9$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13456, 12567, 12456, 12367, 12357]*	$1 \mapsto 7, 2 \mapsto 8, 3 \mapsto 8, 4 \mapsto 7, 5 \mapsto 9, 6 \mapsto 9, 7 \mapsto 8$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13456, 12567, 12456, 12367, 12356]*	$1 \mapsto 7, 2 \mapsto 9, 3 \mapsto 9, 4 \mapsto 8, 5 \mapsto 10, 6 \mapsto 10, 7 \mapsto 9$
[34567, 24567, 23567, 23467, 23457, 23456, 14567, 13567, 13456, 12567, 12456, 12356, 12347]*	$1 \mapsto 6, 2 \mapsto 7, 3 \mapsto 7, 4 \mapsto 7, 5 \mapsto 8, 6 \mapsto 8, 7 \mapsto 7$

Previously-unknown minimal* non-isomorphic generators for FC-families on [8]	
[678, 578, 346, 125]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1, 4 \mapsto 1, 5 \mapsto 2, 6 \mapsto 2, 7 \mapsto 2, 8 \mapsto 2$
[678, 458, 237, 135]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 1, 5 \mapsto 2, 6 \mapsto 1, 7 \mapsto 2, 8 \mapsto 2$
[1578, 678, 458, 237]*	$1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 3, 4 \mapsto 3, 5 \mapsto 4, 6 \mapsto 4, 7 \mapsto 6, 8 \mapsto 6$
[1567, 678, 458, 237]*	$1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 2, 4 \mapsto 2, 5 \mapsto 3, 6 \mapsto 3, 7 \mapsto 4, 8 \mapsto 4$
[1457, 678, 458, 237]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1, 4 \mapsto 2, 5 \mapsto 2, 6 \mapsto 2, 7 \mapsto 3, 8 \mapsto 3$
[45678, 1246, 678, 578, 346]*	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3, 6 \mapsto 7, 7 \mapsto 5, 8 \mapsto 5$
[35678, 2357, 678, 458, 123]*	$1 \mapsto 18, 2 \mapsto 25, 3 \mapsto 30, 4 \mapsto 28, 5 \mapsto 40, 6 \mapsto 27, 7 \mapsto 37, 8 \mapsto 44$
[35678, 1345, 678, 458, 237]*	$1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 5, 4 \mapsto 4, 5 \mapsto 5, 6 \mapsto 4, 7 \mapsto 6, 8 \mapsto 6$
[35678, 1246, 678, 578, 346]*	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 5, 4 \mapsto 5, 5 \mapsto 4, 6 \mapsto 8, 7 \mapsto 6, 8 \mapsto 6$
[34678, 2357, 678, 458, 123]*	$1 \mapsto 8, 2 \mapsto 12, 3 \mapsto 15, 4 \mapsto 16, 5 \mapsto 19, 6 \mapsto 13, 7 \mapsto 18, 8 \mapsto 22$
[34578, 1345, 678, 458, 237]**	$1 \mapsto 2, 2 \mapsto 5, 3 \mapsto 7, 4 \mapsto 5, 5 \mapsto 5, 6 \mapsto 5, 7 \mapsto 9, 8 \mapsto 8$
[34578, 1246, 678, 578, 346]*	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3, 6 \mapsto 7, 7 \mapsto 5, 8 \mapsto 5$
[34568, 1345, 678, 458, 237]*	$1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 6, 4 \mapsto 5, 5 \mapsto 5, 6 \mapsto 6, 7 \mapsto 8, 8 \mapsto 8$
[25678, 1345, 678, 458, 237]*	$1 \mapsto 2, 2 \mapsto 5, 3 \mapsto 6, 4 \mapsto 5, 5 \mapsto 6, 6 \mapsto 5, 7 \mapsto 8, 8 \mapsto 8$
[25678, 1246, 678, 578, 346]*	$1 \mapsto 4, 2 \mapsto 8, 3 \mapsto 11, 4 \mapsto 13, 5 \mapsto 10, 6 \mapsto 20, 7 \mapsto 15, 8 \mapsto 15$
[24678, 1246, 678, 578, 346]*	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3, 6 \mapsto 7, 7 \mapsto 5, 8 \mapsto 5$
[24578, 1345, 678, 458, 237]*	$1 \mapsto 2, 2 \mapsto 7, 3 \mapsto 7, 4 \mapsto 6, 5 \mapsto 6, 6 \mapsto 7, 7 \mapsto 11, 8 \mapsto 10$
[24578, 1246, 678, 578, 346]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 3, 5 \mapsto 2, 6 \mapsto 4, 7 \mapsto 3, 8 \mapsto 3$
[24568, 1345, 678, 458, 237]*	$1 \mapsto 2, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 4, 5 \mapsto 4, 6 \mapsto 5, 7 \mapsto 8, 8 \mapsto 7$
[23678, 1246, 678, 578, 346]*	$1 \mapsto 2, 2 \mapsto 2, 3 \mapsto 5, 4 \mapsto 5, 5 \mapsto 4, 6 \mapsto 8, 7 \mapsto 6, 8 \mapsto 6$
[23567, 1345, 678, 458, 237]*	$1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 5, 4 \mapsto 4, 5 \mapsto 5, 6 \mapsto 5, 7 \mapsto 7, 8 \mapsto 7$
[3456, 1458, 2378, 4678, 2347, 2458]*	$1 \mapsto 2, 2 \mapsto 5, 3 \mapsto 5, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 4, 7 \mapsto 5, 8 \mapsto 6$
[2356, 1568, 3468, 2478, 1268, 1248]*	$1 \mapsto 7, 2 \mapsto 9, 3 \mapsto 6, 4 \mapsto 7, 5 \mapsto 5, 6 \mapsto 9, 7 \mapsto 3, 8 \mapsto 10$
[1357, 1356, 1348, 1346, 1345, 1278, 1268]*	$1 \mapsto 54, 2 \mapsto 26, 3 \mapsto 42, 4 \mapsto 31, 5 \mapsto 30, 6 \mapsto 38, 7 \mapsto 31, 8 \mapsto 36$
[1346, 1345, 1278, 1268, 1267, 1258, 1257, 1256]*	$1 \mapsto 7, 2 \mapsto 6, 3 \mapsto 2, 4 \mapsto 2, 5 \mapsto 5, 6 \mapsto 5, 7 \mapsto 4, 8 \mapsto 4$
[345678, 245678, 235678, 234678, 234578, 234568, 234567, 145678, 135678, 134678, 134578, 134568, 134567, 125678, 124678, 124578, 124568, 124567, 123678, 123578, 123568, 123567, 123478, 123468, 123467, 123456]*	$1 \mapsto 28, 2 \mapsto 28, 3 \mapsto 28, 4 \mapsto 28, 5 \mapsto 28, 6 \mapsto 30, 7 \mapsto 29, 8 \mapsto 29$
[345678, 245678, 235678, 234678, 234578, 234568, 234567, 145678, 135678, 134678, 134578, 134568, 134567, 125678, 124678, 124578, 124568, 124567, 123678, 123578, 123568, 123567, 123478, 123468, 123457, 123456]*	$1 \mapsto 27, 2 \mapsto 27, 3 \mapsto 27, 4 \mapsto 27, 5 \mapsto 28, 6 \mapsto 28, 7 \mapsto 28, 8 \mapsto 28$

Previously-unknown minimal* non-isomorphic generators for FC-families on [9]	
[369, 789, 456, 123]*	$1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 1, 5 \mapsto 1, 6 \mapsto 2, 7 \mapsto 1, 8 \mapsto 1, 9 \mapsto 2$
[348, 569, 789, 1268]*	$1 \mapsto 4, 2 \mapsto 4, 3 \mapsto 8, 4 \mapsto 8, 5 \mapsto 9, 6 \mapsto 11, 7 \mapsto 11, 8 \mapsto 17, 9 \mapsto 16$
[148, 159, 6789, 2345]*	$1 \mapsto 4, 2 \mapsto 1, 3 \mapsto 1, 4 \mapsto 3, 5 \mapsto 3, 6 \mapsto 1, 7 \mapsto 1, 8 \mapsto 3, 9 \mapsto 3$
[589, 129, 6789, 3459]*	$1 \mapsto 18, 2 \mapsto 18, 3 \mapsto 10, 4 \mapsto 10, 5 \mapsto 25, 6 \mapsto 10, 7 \mapsto 10, 8 \mapsto 25, 9 \mapsto 36$
[489, 159, 2345, 5679]*	$1 \mapsto 20, 2 \mapsto 8, 3 \mapsto 8, 4 \mapsto 23, 5 \mapsto 28, 6 \mapsto 10, 7 \mapsto 10, 8 \mapsto 19, 9 \mapsto 34$
[5689, 578, 129, 6789, 3459]*	$1 \mapsto 3, 2 \mapsto 3, 3 \mapsto 2, 4 \mapsto 2, 5 \mapsto 6, 6 \mapsto 3, 7 \mapsto 5, 8 \mapsto 6, 9 \mapsto 6$
[5679, 128, 129, 6789, 3459]*	$1 \mapsto 16, 2 \mapsto 16, 3 \mapsto 3, 4 \mapsto 3, 5 \mapsto 6, 6 \mapsto 7, 7 \mapsto 7, 8 \mapsto 15, 9 \mapsto 17$
[5678, 278, 129, 6789, 3459]*	$1 \mapsto 52, 2 \mapsto 81, 3 \mapsto 16, 4 \mapsto 16, 5 \mapsto 32, 6 \mapsto 35, 7 \mapsto 58, 8 \mapsto 58, 9 \mapsto 75$
[4789, 578, 129, 6789, 3459]*	$1 \mapsto 49, 2 \mapsto 49, 3 \mapsto 34, 4 \mapsto 60, 5 \mapsto 80, 6 \mapsto 36, 7 \mapsto 79, 8 \mapsto 79, 9 \mapsto 98$
[4789, 489, 159, 6789, 2345]*	$1 \mapsto 20, 2 \mapsto 8, 3 \mapsto 8, 4 \mapsto 26, 5 \mapsto 24, 6 \mapsto 11, 7 \mapsto 17, 8 \mapsto 26, 9 \mapsto 36$
[4689, 578, 129, 6789, 3459]*	$1 \mapsto 17, 2 \mapsto 17, 3 \mapsto 12, 4 \mapsto 21, 5 \mapsto 30, 6 \mapsto 19, 7 \mapsto 28, 8 \mapsto 33, 9 \mapsto 35$
[4689, 478, 159, 6789, 2345]*	$1 \mapsto 12, 2 \mapsto 6, 3 \mapsto 6, 4 \mapsto 24, 5 \mapsto 15, 6 \mapsto 14, 7 \mapsto 21, 8 \mapsto 25, 9 \mapsto 21$
[4679, 158, 159, 6789, 2345]*	$1 \mapsto 18, 2 \mapsto 3, 3 \mapsto 3, 4 \mapsto 6, 5 \mapsto 19, 6 \mapsto 7, 7 \mapsto 7, 8 \mapsto 16, 9 \mapsto 17$
[4678, 578, 159, 6789, 2345]*	$1 \mapsto 9, 2 \mapsto 3, 3 \mapsto 3, 4 \mapsto 6, 5 \mapsto 16, 6 \mapsto 6, 7 \mapsto 11, 8 \mapsto 11, 9 \mapsto 12$
[4589, 589, 159, 6789, 2345]*	$1 \mapsto 5, 2 \mapsto 2, 3 \mapsto 2, 4 \mapsto 4, 5 \mapsto 8, 6 \mapsto 2, 7 \mapsto 2, 8 \mapsto 6, 9 \mapsto 8$

Previously-unknown minimal* non-isomorphic generators for FC-families on [10]	
[123, 124, 356, 678, 79(10)]*	$1 \mapsto 6, 2 \mapsto 6, 3 \mapsto 8, 4 \mapsto 4, 5 \mapsto 5, 6 \mapsto 7, 7 \mapsto 5, 8 \mapsto 4, 9 \mapsto 2, 10 \mapsto 2$
[123, 124, 356, 678, 3489(10)]*	$1 \mapsto 7, 2 \mapsto 7, 3 \mapsto 5, 4 \mapsto 5, 5 \mapsto 5, 6 \mapsto 6, 7 \mapsto 3, 8 \mapsto 3, 9 \mapsto 1, 10 \mapsto 1$